Chapter 2

Solutions of Problems

Prob. 2.1

$$\frac{d\phi}{dx} - \frac{1}{Pe}\frac{d^2\phi}{dx^2} = 0 \quad \forall \; x \in \Omega = (0,1) \subset R^1 \qquad \text{——} (1)$$

$$\phi(0) = 1, \quad \phi(1) = 0 \qquad \text{——} (2)$$

Solution:

a) Differential operator in (1)

$$A = \frac{d}{dx} - \frac{1}{Pe}\frac{d^2}{dx^2} \qquad \text{——} (3)$$

∴ (1) can be written as

$$A\phi = 0 \; ; \; x \in \Omega = (0,1) \subset R^1 \qquad \text{——} (4)$$

b) Linearity of A:

$\forall \; u, v \in D_A$, domain of definition of A

and $\forall \; \alpha, \beta \in R$

If we can show that

$A(\alpha u + \beta v) = \alpha Au + \beta Av$ Then A is linear —— (5)

Consider $A(\alpha u + \beta v)$

$$A(\alpha u + \beta v) = \left(\frac{d}{dx} - \frac{1}{Pe}\frac{d^2}{dx^2}\right)(\alpha u + \beta v)$$

$$= \left(\frac{d}{dx} - \frac{1}{Pe}\frac{d^2}{dx^2}\right)(\alpha u) + \left(\frac{d}{dx} - \frac{1}{Pe}\frac{d^2}{dx^2}\right)(\beta v)$$

using product rule of differentiation for αu and βv.

$$A(\alpha u + \beta v) = \alpha\left(\frac{d}{dx} - \frac{1}{Pe}\frac{d^2}{dx^2}\right)u + \beta\left(\frac{d}{dx} - \frac{1}{Pe}\frac{d^2}{dx^2}\right)v$$

$$= \alpha Au + \beta Av \qquad \text{——} (6)$$

Hence in this case

$A(\alpha u + \beta v) = \alpha Au + \beta Av$ holds.

∴ The differential operator A is linear.

Prob 2.2

$$\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) = 0 \quad \forall x \in \Omega = (0,1) \subset R^1 \quad \text{——— (1)}$$

$$\phi(0) = 0 \quad \left(a(x)\frac{d\phi}{dx}\right)\Big|_1 = P_1 \quad \text{——— (2)}$$

Solution :

(a) differential operator in (1)

$$A = \frac{d}{dx}\left(a(x)\frac{d}{dx}\right) \quad \text{——— (3)}$$

$\therefore$ (1) can be written as

$$A\phi = 0, \ \forall x \in \Omega = (0,1) \subset R^1 \quad \text{——— (4)}$$

(b) Linearity of A :

$\forall u, v \in D_A$

and $\forall \alpha, \beta \in R$

If we can show that

$A(\alpha u + \beta v) = \alpha Au + \beta Av$ then A is linear ——— (5)

Consider $A(\alpha u + \beta v)$

$$A(\alpha u + \beta v) = \frac{d}{dx}\left(a(x)\frac{d}{dx}\right)(\alpha u + \beta v)$$

$$= \left(\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)\right)(\alpha u) + \left(\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)\right)(\beta v)$$

using product rule of differentiation for αu and βv

$$A(\alpha u + \beta v) = \alpha\left(\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)\right)u + \beta\left(\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)\right)v$$

$$= \alpha Au + \beta v \quad \text{——— (6)}$$

Hence, in this case

$A(\alpha u + \beta v) = \alpha Au + \beta Av$ holds

$\therefore$ The differential operator A is linear

Prob. 2.3

$$\phi \frac{d\phi}{dx} - \frac{1}{Re}\frac{d^2\phi}{dx^2} = 0 \quad \forall\, x \in \Omega = (0,1) \subset R^1 \quad \text{—— (1)}$$

$$\phi(0) = 1 \;; \quad \phi(1) = 0 \quad \text{—— (2)}$$

Solution:

(a) differential operator A:

$$A = \phi \frac{d}{dx} - \frac{1}{Re}\frac{d^2}{dx^2} \quad \text{—— (3)}$$

∴ (1) can be written as

$$A\phi = 0 \quad \forall\, x \in \Omega = (0,1) \subset R^1 \quad \text{—— (4)}$$

(b) linearity of A:

$\forall\, u, v \in D_A$

and $\forall\, \alpha, \beta \in R$

If we can show that

$A(\alpha u + \beta v) = \alpha A u + \beta A v$ then A is linear —— (5)

Consider $A(\alpha u + \beta v)$; substitute $\alpha u + \beta v$ for ϕ in (4) and (3)

$$A(\alpha u + \beta v) = \left((\alpha u + \beta v)\frac{d}{dx} - \frac{1}{Re}\frac{d^2}{dx^2}\right)(\alpha u + \beta v)$$

$$= \alpha(\alpha u + \beta v)\frac{du}{dx} + \beta(\alpha u + \beta v)\frac{dv}{dx} - \frac{\alpha}{Re}\frac{d^2u}{dx^2} - \frac{\beta}{Re}\frac{d^2v}{dx^2} \quad \text{—— (6)}$$

on the other hand (right side of (5))

$$\alpha A u + \beta A v = \alpha\left(u\frac{d}{dx} - \frac{1}{Re}\frac{d^2}{dx^2}\right)u + \beta\left(v\frac{d}{dx} - \frac{1}{Re}\frac{d^2}{dx^2}\right)v$$

$$= \alpha u \frac{du}{dx} + \beta v \frac{dv}{dx} - \frac{\alpha}{Re}\frac{d^2u}{dx^2} - \frac{\beta}{Re}\frac{d^2v}{dx^2} \quad \text{—— (7)}$$

Even though the last two terms in (6) and (7) are exactly same, but the first two are not, hence

$A(\alpha u+\beta v) = \alpha A u + \beta A v$ does not hold in this case. Thus, the differential operator A in (3) is not linear.

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Prob. 2.4

$$B(\phi, v) = \int_1^2 a(x)\frac{d\phi}{dx}\frac{dv}{dx}dx + b(2)\phi(2)v(2) \quad \text{—(1)}$$

(a) Bilinearity of $B(\phi, v)$

If $B(\phi, v)$ is bilinear, then it must be linear in each one of its arguments i.e linear in ϕ as well as linear in v.

I. linearity in ϕ

Let $u, w \in D_B$

and $\alpha, \beta \in R$

Then we must show that

$$B(\alpha u+\beta w, v) = \alpha B(u,v) + \beta B(w,v) \text{ holds} \quad \text{—(2)}$$

if $B(\phi, v)$ is linear in ϕ.

Consider

$$B(\alpha u+\beta w, v) = \int_1^2 a(x)\frac{d(\alpha u+\beta w)}{dx}\frac{dv}{dx}dx + b(2)\left(\alpha u(2)+\beta w(2)\right)v(2) \quad \text{—(3)}$$

Expanding right side and regrouping the terms.

$$B(\alpha u+\beta w, v) = \alpha\left(\int_1^2 a(x)\,\frac{du}{dx}\frac{dv}{dx}\,dx + b(2)\,u(2)\,v(2)\right) +$$

$$\beta\left(\int_1^2 a(x)\,\frac{dw}{dx}\frac{dv}{dx}\,dx + b(2)\,w(2)\,v(2)\right)$$

$$= \alpha\, B(u,v) + \beta\,(w,v) \quad \text{——— (4)}$$

Hence, (2) holds, thus $B(\phi, v)$ is linear in ϕ.

II. Linearity in v

Let $v_1, v_2 \in D_B$

and $\alpha_1, \alpha_2 \in R$

Then we must show that

$$B(\phi,\ \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1\, B(\phi, v_1) + \alpha_2\, B(\phi, v_2) \text{ holds} \quad \text{— (5)}$$

if $B(\phi, v)$ is linear in v.

Consider

$$B(\phi,\ \alpha_1 v_1 + \alpha_2 v_2) = \int_1^2 a(x)\,\frac{d\phi}{dx}\frac{d}{dx}(\alpha_1 v_1 + \alpha_2 v_2)\,dx + b(2)\,\phi(2)\big(\alpha_1 v_1(2) + \alpha_2 v_2(2)\big) \quad \text{——— (6)}$$

Expanding the right side of (6) and rearranging the terms

$$B(\phi,\ \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1\left(\int_1^2 a(x)\,\frac{d\phi}{dx}\frac{dv_1}{dx}\,dx + b(2)\,\phi(2)\,v_1(2)\right) +$$

$$\alpha_2\left(\int_1^2 a(x)\,\frac{d\phi}{dx}\frac{dv_2}{dx}\,dx + b(2)\,\phi(2)\,v_2(2)\right)$$

$$= \alpha_1\, B(\phi, v_1) + \alpha_2\, B(\phi, v_2) \quad \text{——— (7)}$$

Hence, (5) holds, thus $B(\phi, v)$ is linear in v.

Thus, $B(\phi, v)$ is linear in ϕ as well v, hence

$B(\phi, v)$ is bilinear.

(b) Symmetry of $B(\phi, v)$

If $B(\phi, v)$ is symmetric then

$$B(\phi, v) = B(v, \phi) \text{ must hold.} \qquad (8)$$

$$B(\phi, v) = \int_1^2 a(x) \frac{d\phi}{dx}\frac{dv}{dx}\,dx + b(2)\,\phi(2)\,v(2) \qquad (9)$$

$$B(v, \phi) = \int_1^2 a(x) \frac{dv}{dx}\frac{d\phi}{dx}\,dx + b(2)\,v(2)\,\phi(2) \qquad (10)$$

$$= \int_1^2 a(x) \frac{d\phi}{dx}\frac{dv}{dx}\,dx + b(2)\,\phi(2)\,v(2) \qquad (11)$$

Comparing (9) and (11) it is straight forward to conclude that

$B(\phi, v) = B(v, \phi)$ holds

Hence $B(\phi, v)$ is symmetric.

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Prob. 2.5

$$B(\phi, v) = \int_{-1}^{1} a(x) \frac{d^2\phi}{dx^2} \frac{d^2 v}{dx^2}\, dx + \int_{-1}^{1} c(x)\, \phi\, v\, dx \quad \text{—— (1)}$$

(a) Bilinearity of $B(\phi, v)$

If $B(\phi, v)$ is bilinear, then it must be linear in each one of its arguments, i.e linear in ϕ as well as linear in v.

I. Linearity in ϕ

Let $\phi_1, \phi_2 \in D_B$

and $\alpha_1, \alpha_2 \in R$

Then we must show that

$$B(\alpha_1\phi_1 + \alpha_2\phi_2, v) = \alpha_1 B(\phi_1, v) + \alpha_2 B(\phi_2, v) \text{ holds} \quad \text{—— (2)}$$

if $B(\phi, v)$ is linear in ϕ.

Consider

$$B(\alpha_1\phi_1 + \alpha_2\phi_2, v) = \int_{-1}^{1} a(x) \frac{d^2(\alpha_1\phi_1 + \alpha_2\phi_2)}{dx^2} \frac{d^2 v}{dx^2}\, dx + \int_{-1}^{1} c(x)\,(\alpha_1\phi_1 + \alpha_2\phi_2)\, v\, dx \quad \text{—— (3)}$$

Expanding right side and rearranging the terms

$$B(\alpha_1\phi_1 + \alpha_2\phi_2, v) = \alpha_1 \left(\int_{-1}^{1} a(x) \frac{d^2\phi_1}{dx^2} \frac{d^2 v}{dx^2}\, dx + \int_{-1}^{1} c(x)\,\phi_1 v\, dx \right) + \alpha_2 \left(\int_{-1}^{1} a(x) \frac{d^2\phi_2}{dx^2} \frac{d^2 v}{dx^2}\, dx + \int_{-1}^{1} c(x)\,\phi_2 v\, dx \right)$$

$$= \alpha_1 B(\phi_1, v) + \alpha_2 B(\phi_2, v) \quad \text{—— (4)}$$

Hence, (2) holds, thus $B(\phi, v)$ is linear in ϕ.

II Linearity in v

Let $v_1, v_2 \in D_B$

and $\beta_1, \beta_2 \in R$

Then we must show that

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \text{ holds} \quad \text{--- (5)}$$

if $B(\phi, v)$ is linear in v.

Consider

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \int_{-1}^{1} a(x) \frac{d^2\phi}{dx^2} \frac{d^2(\beta_1 v_1 + \beta_2 v_2)}{dx^2} dx + \int_{-1}^{1} c(x)\, \phi \,(\beta_1 v_1 + \beta_2 v_2)\, dx \quad \text{--- (6)}$$

Expanding the right side of (6) and rearranging the terms.

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \beta_1 \left(\int_{-1}^{1} a(x) \frac{d^2\phi}{dx^2} \frac{d^2 v_1}{dx^2} dx + \int_{-1}^{1} c(x)\, \phi\, v_1\, dx \right) + \beta_2 \left(\int_{-1}^{1} a(x) \frac{d^2\phi}{dx^2} \frac{d^2 v_2}{dx^2} dx + \int_{-1}^{1} c(x)\, \phi\, v_2\, dx \right)$$

$$= \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \quad \text{--- (7)}$$

Hence, (5) holds, thus $B(\phi, v)$ is linear in v.

Thus, $B(\phi, v)$ is linear in ϕ as well as v, hence $B(\phi, v)$ is bilinear.

(b) Symmetry of $B(\phi, v)$

If $B(\phi, v)$ is symmetric, then

$$B(\phi, v) = B(v, \phi) \text{ must hold.} \qquad \text{—— } (8)$$

$$B(\phi, v) = \int_{-1}^{1} a(x) \frac{d^2\phi}{dx^2} \frac{d^2 v}{dx^2} dx + \int_{-1}^{1} c(x)\, \phi v \, dx \qquad \text{—— } (9)$$

$$B(v, \phi) = \int_{-1}^{1} a(x) \frac{d^2 v}{dx^2} \frac{d^2\phi}{dx^2} dx + \int_{-1}^{1} c(x)\, v \phi \, dx \qquad \text{—— } (10)$$

$$= \int_{-1}^{1} a(x) \frac{d^2\phi}{dx^2} \frac{d^2 v}{dx^2} dx + \int_{-1}^{1} c(x)\, \phi v \, dx \qquad \text{—— } (11)$$

Comparing (9) and (11) we conclude that:

$B(\phi, v) = B(v, \phi)$ holds

Hence $B(\phi, v)$ is symmetric.

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Prob. 2.6

$$B(\phi, v) = \int_{-1}^{1} \phi \frac{d\phi}{dx}\frac{dv}{dx}\,dx \quad \text{——— (1)}$$

(a) Bilinearity of $B(\phi, v)$

If $B(\phi, v)$ is bilinear, then it must be linear in each one of its arguments i.e linear in ϕ as well as linear in v.

I. Linearity in ϕ

Let $\phi_1, \phi_2 \in D_B$

and $\alpha_1, \alpha_2 \in R$

Then we must show that

$$B(\alpha_1\phi_1 + \alpha_2\phi_2, v) = \alpha_1 B(\phi_1, v) + \beta_2 B(\phi_2, v) \text{ holds} \quad \text{— (2)}$$

if $B(\phi, v)$ is linear in ϕ.

Consider

$$B(\alpha_1\phi_1 + \alpha_2\phi_2, v) = \int_{-1}^{1} (\alpha_1\phi_1 + \alpha_2\phi_2)\frac{d}{dx}(\alpha_1\phi_1 + \alpha_2\phi_2)\frac{dv}{dx}\,dx \quad \text{— (3)}$$

$$= \alpha_1 \int_{-1}^{1} \phi_1 \frac{d}{dx}(\alpha_1\phi_1 + \alpha_2\phi_2)\frac{dv}{dx}\,dx \; +$$

$$\alpha_2 \int_{-1}^{1} \phi_2 \frac{d}{dx}(\alpha_1\phi_1 + \alpha_2\phi_2)\frac{dv}{dx}\,dx$$

$$= \alpha_1^2 \int_{-1}^{1} \phi_1 \frac{d\phi_1}{dx}\frac{dv}{dx}\,dx + \alpha_1\alpha_2 \int_{-1}^{1} \phi_1 \frac{d\phi_2}{dx}\frac{dv}{dx}\,dx \; +$$

$$\alpha_1\alpha_2 \int_{-1}^{1} \phi_2 \frac{d\phi_1}{dx}\frac{dv}{dx}\,dx + \alpha_2^2 \int_{-1}^{1} \phi_2 \frac{d\phi_2}{dx}\frac{dv}{dx}\,dx \quad \text{— (4)}$$

Consider

$$\alpha_1 B(\phi_1, v) = \alpha_1 \int_{-1}^{1} \phi_1 \frac{d\phi_1}{dx}\frac{dv}{dx}\,dx \quad \text{——— (5)}$$

and $$\alpha_2 B(\phi_2, v) = \alpha_2 \int_{-1}^{1} \phi_2 \frac{d\phi_2}{dx}\frac{dv}{dx}\,dx \quad \text{——— (6)}$$

From (4), (5) and (6) we note that (2) does not hold, thus $B(\phi, v)$ is not linear in ϕ.

II. Linearity in v

Let $v_1, v_2 \in D_B$

and $\beta_1, \beta_2 \in R$

Then, we must show that

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \text{ holds} \quad \text{—— (7)}$$

if $B(\phi, v)$ is linear in v.

Consider

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \int_{-1}^{1} \phi \frac{d\phi}{dx}\frac{d}{dx}(\beta_1 v_1 + \beta_2 v_2)\,dx$$

$$= \beta_1 \int_{-1}^{1} \phi \frac{d\phi}{dx}\frac{dv_1}{dx}\,dx + \beta_2 \int_{-1}^{1} \phi \frac{d\phi}{dx}\frac{dv_2}{dx}\,dx$$

$$= \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \quad \text{—— (8)}$$

Hence, (7) holds, thus $B(\phi, v)$ is linear in v.

From I. and II. we note that $B(\phi, v)$ is linear in v but not linear in ϕ, hence $B(\phi, v)$ is not bilinear.

(b) Symmetry of $B(\phi, v)$

If $B(\phi, v)$ is symmetric then
$B(\phi, v) = B(v, \phi)$ must hold. —— (9)

- Since $B(\phi, v)$ is not bilinear, it can not be symmetric, but we consider details in the following.

$$B(\phi, v) = \int_{-1}^{1} \phi \frac{d\phi}{dx}\frac{dv}{dx}\,dx \qquad \text{—— (10)}$$

$$B(v, \phi) = \int_{-1}^{1} v \frac{dv}{dx}\frac{d\phi}{dx}\,dx \qquad \text{—— (11)}$$

Clearly (10) and (11) are not the same, Thus $B(\phi, v)$ is not symmetric

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Prob. 2.7

$$B(\phi, v) = \int_{-1}^{1} \frac{d\phi}{dx} v\,dx + \frac{1}{Pe}\int_{-1}^{1} \frac{d\phi}{dx}\frac{dv}{dx}\,dx \qquad \text{—— (1)}$$

(a) Bilinearity of $B(\phi, v)$

If $B(\phi, v)$ is bilinear, then it must be linear in each one of its arguments i,e linear in ϕ as well as linear in v.

I. Linearity in ϕ

Let $\phi_1, \phi_2 \in D_B$

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and $\alpha_1, \alpha_2 \in R$

Then we must show that

$$B(\alpha_1\phi_1+\alpha_2\phi_2, v) = \alpha_1 B(\phi_1, v) + \alpha_2 B(\phi_2, v) \text{ holds} \quad \text{— (2)}$$

if $B(\phi, v)$ is linear in ϕ.

Consider

$$B(\alpha_1\phi_1+\alpha_2\phi_2, v) = \int_{-1}^{1} \frac{d}{dx}(\alpha_1\phi_1+\alpha_2\phi_2)\, v\, dx + \frac{1}{Pe}\int_{-1}^{1} \frac{d}{dx}(\alpha_1\phi_1+\alpha_2\phi_2) \frac{dv}{dx} dx$$

$$= \alpha_1\left(\int_{-1}^{1}\frac{d\phi_1}{dx} v\, dx + \frac{1}{Pe}\int_{-1}^{1}\frac{d\phi_1}{dx}\frac{dv}{dx}dx\right) + \alpha_2\left(\int_{-1}^{1}\frac{d\phi_2}{dx} v\, dx + \frac{1}{Pe}\int_{-1}^{1}\frac{d\phi_2}{dx}\frac{dv}{dx}dx\right)$$

$$= \alpha_1 B(\phi_1, v) + \alpha_2 B(\phi_2, v) \quad \text{— (3)}$$

Hence (2) holds, thus $B(\phi, v)$ is linear in ϕ.

(b) Linearity in v

Let $v_1, v_2 \in D_B$

and $\beta_1, \beta_2 \in R$

Then we must show that

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \quad \text{— (4)}$$

must hold if $B(\phi, v)$ is linear in v.

Consider

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \int_{-1}^{1}\frac{d\phi}{dx}(\beta_1 v_1 + \beta_2 v_2)\,dx + \frac{1}{Re}\int_{-1}^{1}\frac{d\phi}{dx}\frac{d}{dx}(\beta_1 v_1 + \beta_2 v_2)\,dx \quad \text{— (5)}$$

or

$$B(\phi, \beta_1 v_1 + \beta_2 v_2) = \beta_1 \left(\int_{-1}^{1} \frac{d\phi}{dx} v_1 \, dx + \frac{1}{Pe} \int_{-1}^{1} \frac{d\phi}{dx} \frac{dv_1}{dx} dx \right) +$$

$$\beta_2 \left(\int_{-1}^{1} \frac{d\phi}{dx} v_2 \, dx + \frac{1}{Pe} \int_{-1}^{1} \frac{d\phi}{dx} \frac{dv_2}{dx} dx \right)$$

$$= \beta_1 B(\phi, v_1) + \beta_2 B(\phi, v_2) \quad \text{—— (6)}$$

Hence, (4) holds, thus $B(\phi, v)$ is linear in v.

Thus, $B(\phi, v)$ is linear in ϕ as well as in v, hence $B(\phi, v)$ is bilinear.

(b) Symmetry of $B(\phi, v)$

If $B(\phi, v)$ is symmetric, then

$$B(\phi, v) = B(v, \phi) \text{ must hold} \quad \text{—— (7)}$$

$$B(\phi, v) = \int_{-1}^{1} \frac{d\phi}{dx} v \, dx + \frac{1}{Pe} \int_{-1}^{1} \frac{d\phi}{dx} \frac{dv}{dx} dx \quad \text{—— (8)}$$

$$B(v, \phi) = \int_{-1}^{1} \frac{dv}{dx} \phi \, dx + \frac{1}{Pe} \int_{-1}^{1} \frac{dv}{dx} \frac{d\phi}{dx} dx$$

$$= -\int_{-1}^{1} \phi \frac{dv}{dx} dx + \frac{1}{Pe} \int_{-1}^{1} \frac{d\phi}{dx} \frac{dv}{dx} dx \quad \text{—— (9)}$$

From (8) and (9), we conclude that

$$B(\phi, v) \neq B(v, \phi) \quad \text{—— (10)}$$

Hence $B(\phi, v)$ is not symmetric.

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Prob 2.8

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + cu = q(x) \quad \forall x \in \Omega = (0,1) \subset R^1 \quad \text{---(1)}$$

$$\text{BCs:} \quad u(0) = u_0 \; ; \; \left(a(x)\frac{du}{dx} + \beta(u - u_\infty)\right)\Big|_{x=1} = Q_1 \quad \text{---(2)}$$

(a) Differential operator

$$A = -\frac{d}{dx}\left(a(x)\frac{d}{dx}\right) + c \quad \text{---(3)}$$

$$\therefore \quad Au - f = 0 \quad \forall x \in \Omega = (0,1) \subset R^1 \quad \text{---(4)}$$

where $f = q(x)$

(b) linearity of A :

Let $u_1, u_2 \in D_A$

and $\alpha_1, \alpha_2 \in R$

Then

$$A(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 A u_1 + \alpha_2 A u_2 \quad \text{must hold} \quad \text{---(5)}$$

If the operator A is linear

Consider $A(\alpha_1 u_1 + \alpha_2 u_2) = \left(-\frac{d}{dx}\left(a(x)\frac{d}{dx}\right) + c\right)(\alpha_1 u_1 + \alpha_2 u_2)$

$$= \left(-\frac{d}{dx}\left(a(x)\frac{d}{dx}\right) + c\right)(\alpha_1 u_1) + \left(-\frac{d}{dx}\left(a(x)\frac{d}{dx}\right) + c\right)(\alpha_2 u_2)$$

$$= \alpha_1\left(-\frac{d}{dx}\left(a(x)\frac{du_1}{dx}\right) + cu_1\right) + \alpha_2\left(-\frac{d}{dx}\left(a(x)\frac{du_2}{dx}\right) + cu_2\right)$$

$\therefore A(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 A u_1 + \alpha_2 A u_2$ —— (6)

Based on (4), (5) holds, hence the operator A is linear.

(c) <u>Symmetry of A.</u>

$\forall\ u, v \in D_A$ with $v = \delta u$

We must show that

$(Au, v) = (u, Av)$ —— (6)

if A is symmetric

Consider (Au, v)

$$(Au, v) = \int_0^1 \left(-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + cu\right) v\, dx$$

$$= \int_0^1 -\frac{d}{dx}\left((a)\frac{du}{dx}\right) v\, dx + \int_0^1 cuv\, dx \quad \text{—— (7)}$$

Perform integration by parts twice in the first term to transfer all differentiation from u to v.

IBP once give:

$$(Au, v) = \int_0^1 \left(\frac{dv}{dx} a(x)\right)\frac{du}{dx}\, dx - \left(\left(a(x)\frac{du}{dx}\right)v\right)\Big|_0^1 + \int_0^1 cuv\, dx \quad \text{—— (8)}$$

IBP once more (first term) and regroup terms.

$$(Au, v) = \int_0^1 \left(-\frac{d}{dx}\left(a(x)\frac{dv}{dx}\right) + cv\right) u\, dx + \left(\left(a(x)\frac{dv}{dx}\right)u\right)\Big|_0^1 -$$

$$\left(\left(a(x)\frac{du}{dx}\right)v\right)\Big|_0^1 \qquad \text{——— (9)}$$

or

$$(Au, v) = (u, A^*v) + \langle Au, v\rangle_\Gamma \qquad \text{——— (10)}$$

where $$A^* = -\frac{d}{dx}\left(a(x)\frac{d}{dx}\right) + c \qquad \text{——— (11)}$$

and $$\langle Au, v\rangle_\Gamma = \left(a(x)\frac{dv}{dx}u\right)\Big|_0^1 - \left(a(x)\frac{du}{dx}v\right)\Big|_0^1 \qquad \text{——— (12)}$$

Simplification of Concomitant:

Consider Concomitant $\langle Au, v\rangle$ in (12). Expanding terms

$$\langle Au, v\rangle_\Gamma = a(1)\,u(1)\frac{dv}{dx}\Big|_1 - a(0)\,u(0)\frac{dv}{dx}\Big|_0$$

$$- a(1)\,v(1)\frac{du}{dx}\Big|_1 + a(0)\,v(0)\frac{du}{dx}\Big|_0 \qquad \text{——— (13)}$$

Recall BCs in (2):

$$u(0) = 0 \Rightarrow v(0) = 0 \qquad \text{——— (14)}$$

$$\left(a(x)\frac{du}{dx}\right)\Big|_1 = Q_1 - \beta(1)\left(u(1) - u_\infty(1)\right) \qquad \text{——— (15)}$$

$$\therefore\ a(1)\frac{dv}{dx}\Big|_1 = -\beta(1)\,v(1) \qquad \text{——— (16)}$$

using (14), (15) and (16) in (13)

$$\langle Au, v\rangle_\Gamma = u(1)\left(-\beta(1)\,v(1)\right) - a(0)\,u_0\,\frac{du}{dx}\Big|_0$$

$$- v(1)\left(Q_1 - \beta(1)\left(u(1) - u_\infty(1)\right)\right) + a(0)\, \cancel{v(0)}^{\,0} \left.\frac{du}{dx}\right|_0$$

$$\therefore \langle Au, v\rangle_M = - \beta(1)\cancel{u(1)}v(1) - \left(a(0)\left.\frac{du}{dx}\right|_0\right)u_0$$

$$- v(1)\,Q_1 + \beta(1)\cancel{u(1)}\,v(1) - \beta(1)\,v(1)\,u_\infty(1)$$

$$\therefore \langle Au, v\rangle_M = - \left(a(0)\left.\frac{du}{dx}\right|_0\right)u_0 - v(1)\,Q_1 - \left(\beta(1)\,v(1)\right)u_\infty(1) \qquad (17)$$

Hence, finally we have

$$(Au, v) = (u, A^* v) \underbrace{- \left(a(0)\left.\frac{du}{dx}\right|_0\right)u_0 - v(1)Q_1 - \beta(1)v(1)\,u_\infty(1)}_{\langle Au, v\rangle_M} \qquad (18)$$

where

$$A^* = -\frac{d}{dx}\left(a(x)\frac{d}{dx}\right) + c = A \qquad (19)$$

Concomitant $\langle Au, v\rangle$ in (17) is not zero if u_0, Q_1 and $u_\infty(1)$ are not zero i.e if the BCs are not homogeneous.

- Thus if u_0, Q_1 and $u_\infty(1)$ are not zero, the differential operator A is not symmetric

- A is symmetric if $u_0 = 0$, $Q_1 = 0$ and $u_\infty(1) = 0$ i.e if the BCs are homogeneous.

———— o ————

Prob 2.9

$$-\frac{d}{dx}\left(b(x)\frac{d\phi}{dx}\right) = q(x) \quad \forall\, x \in \Omega = (0,L) \subset R^1 \qquad -(1)$$

BCs: $\phi(0) = 0$, $\left.\left(b(x)\frac{d\phi}{dx} + k\phi\right)\right|_{x=L} = P_L$ —— (2)

(a) Differential operator

$$A = -\frac{d}{dx}\left(b(x)\frac{d}{dx}\right) \qquad \text{—— (3)}$$

$$A\phi - f = 0 \; ; \; f = q \quad \forall\, x \in (0,L) \subset R^1 \qquad \text{— (4)}$$

(b) Linearity of A

Let $\phi_1, \phi_2 \in D_A$

and $\alpha_1, \alpha_2 \in R$

Then $A(\alpha_1\phi_1 + \alpha_2\phi_2) = \alpha_1 A\phi_1 + \alpha_2 A\phi_2$ must hold — (5)

If the operator A is linear.

Consider
$$A(\alpha_1\phi_1 + \alpha_2\phi_2) = \left(-\frac{d}{dx}\left(b(x)\frac{d}{dx}\right)\right)(\alpha_1\phi_1 + \alpha_2\phi_2)$$
$$= \left(-\frac{d}{dx}\left(b(x)\frac{d}{dx}\right)\right)(\alpha_1\phi_1) + \left(-\frac{d}{dx}\left(b(x)\frac{d}{dx}\right)\right)(\alpha_2\phi_2)$$
$$= \alpha_1\left(-\frac{d}{dx}\left(b(x)\frac{d\phi_1}{dx}\right)\right) + \alpha_2\left(-\frac{d}{dx}\left(b(x)\frac{d\phi_2}{dx}\right)\right)$$
$$= \alpha_1 A\phi_1 + \alpha_2 A\phi_2 \qquad \text{—— (6)}$$

Based on (6), (5) holds, hence the operator A is linear.

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(c) Symmetry of A

$\forall\, \phi, v \in D_A$ with $v = \delta\phi$

we must show that

$$(A\phi, v) = (\phi, Av) \qquad \text{——— (7)}$$

if A is symmetric

Consider $(A\phi, v)$

$$(A\phi, v) = \int_0^L \left(-\frac{d}{dx}\left(b(x)\frac{d\phi}{dx}\right)\right) v\, dx \qquad \text{——— (8)}$$

IBP once : transfer one order of differentiation from $b(x)\frac{d\phi}{dx}$ to v

$$(A\phi, v) = \int_0^L \left(\frac{dv}{dx} b(x)\right)\frac{d\phi}{dx}\, dx - \left(b(x)\frac{d\phi}{dx} v\right)\Big|_0^L \qquad \text{——— (9)}$$

IBP once more : transfer on order of differentiation from ϕ to $b(x)\frac{dv}{dx}$

$$(A\phi, v) = \int_0^L -\frac{d}{dx}\left(b(x)\frac{dv}{dx}\right)\phi\, dx + \left(\frac{dv}{dx} b(x)\phi\right)\Big|_0^L - \left(b(x)\frac{d\phi}{dx} v\right)\Big|_0^L \qquad \text{——— (10)}$$

$$\therefore\ (A\phi, v) = (\phi, A^* v) + \langle A\phi, v\rangle_\mu \qquad \text{——— (11)}$$

In which

$$A^* = -\frac{d}{dx}\left(b(x)\frac{d}{dx}\right) \qquad \text{——— (12)}$$

$$\langle A\phi, v\rangle_\mu = \left(\frac{dv}{dx} b(x)\phi\right)\Big|_0^L - \left(b(x)\frac{d\phi}{dx} v\right)\Big|_0^L \qquad \text{——— (13)}$$

Simplification of Concomitant

Expanding the boundary terms in (13)

$$\langle A\phi, v\rangle_\mu = \left(\frac{dv}{dx}\Big|_L b(L)\right)\phi(L) - \frac{dv}{dx}\Big|_0 b(0)\phi(0) - b(L)\frac{d\phi}{dx}\Big|_L v(L) + b(0)\frac{d\phi}{dx}\Big|_0 v(0) \qquad \text{——— (14)}$$

Recall BCs in (2):

$$\left.\begin{aligned} \phi(0) = 0 \quad &\Rightarrow \quad v(0) = 0 \\ \left(b(x)\frac{d\phi}{dx}\right)\Big|_L = P_L - (k\phi)\Big|_L \quad &\Rightarrow \quad b(L)\frac{dv}{dx}\Big|_L = -k\,v(L) \end{aligned}\right\} \text{—(15)}$$

Consider Concomitant (14) and substitute from

$$\langle A\phi, v\rangle_M = \left(-k\,v(L)\right)\phi(L) - \frac{dv}{dx}\Big|_0 b(0)\,\cancelto{0}{\phi(0)}$$

$$- \left(P_L - k\,\phi(L)\right)v(L) + b(0)\frac{d\phi}{dx}\Big|_0 \cancelto{0}{v(0)}$$

$$- k v(L)\phi(L) + k\,v(L)\phi(L) - P_L v(L) \quad \text{—(16)}$$

$$\therefore \langle A\phi, v\rangle_M = -P_L v(L) \quad \text{—— (17)}$$

i.e. Concomitant $\langle A\phi, v\rangle_M \neq 0$ if $P_L \neq 0$ i.e $\langle A\phi, v\rangle_M$ is only zero if the BCs are homogeneous

Adjoint of A

$$A^* = -\frac{d}{dx}\left(b(x)\frac{d}{dx}\right) = A \quad \text{—— (18)}$$

based on (17) and (18) The differential operator A is symmetric only if $P_L = 0$ i.e The BCs are homogeneous.

Prob 2.10

$$\frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi}{dx^2}\right) + \frac{d}{dx}\left(c(x)\frac{d\psi}{dx}\right) + d(x)\psi = f(x) \qquad (1)$$

$$\forall x \in \Omega = (0, L) \subset R^1$$

BCs:

$$\left.\begin{aligned} &\psi(0) = \psi_0 \\ &\psi(L) = \psi_L \\ &\left(b(x)\frac{d^2\psi}{dx^2}\right)\Big|_{x=0} = M_0 \\ &\left(b(x)\frac{d^2\psi}{dx^2}\right)\Big|_{x=L} = M_L \end{aligned}\right\} \qquad (2)$$

(a) differential operator A

$$A = \frac{d^2}{dx^2}\left(b(x)\frac{d^2}{dx^2}\right) + \frac{d}{dx}\left(c(x)\frac{d}{dx}\right) + d(x) \qquad (3)$$

$$\therefore \quad A\psi - f = 0 \quad \forall x \in (0, L) \subset R^1 \qquad (4)$$

(b) Linearity of A

Let $\psi_1, \psi_2 \in D_A$

and $\alpha_1, \alpha_2 \in R$

Then

$$A(\alpha_1\psi_1 + \alpha_2\psi_2) = \alpha_1 A\psi_1 + \alpha_2 A\psi_2 \text{ must hold} \qquad (5)$$

If the operator A is linear.

Consider
$$A(\alpha_1\psi_1 + \alpha_2\psi_2) = \left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2}{dx^2}\right) + \frac{d}{dx}\left(c(x)\frac{d}{dx}\right) + d(x)\right)(\alpha_1\psi_1 + \alpha_2\psi_2) \qquad (5)$$

or

$$A(\alpha_1 \psi_1 + \alpha_2 \psi_2) = \left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2}{dx^2}\right) + \frac{d}{dx}\left(c(x)\frac{d}{dx}\right) + d(x)\right)(\alpha_1 \psi_1) +$$
$$\left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2}{dx^2}\right) + \frac{d}{dx}\left(c(x)\frac{d}{dx}\right) + d(x)\right)(\alpha_2 \psi_2)$$

$$= \alpha_1 \left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi_1}{dx^2}\right) + \frac{d}{dx}\left(c(x)\frac{d\psi_1}{dx}\right) + d(x)\psi_1\right) +$$
$$\alpha_2 \left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi_2}{dx^2}\right) + \frac{d}{dx}\left(c(x)\frac{d\psi_2}{dx}\right) + d(x)\psi_2\right)$$

$$= \alpha_1 A\psi_1 + \alpha_2 A\psi_2 \quad \text{———} \quad (6)$$

Based on (6), (5) holds, hence the operator A is linear.

(c) Symmetry of operator A

$\forall\ \psi, v \in D_A$ with $v = \delta\psi$

we must show that

$$(A\psi, v) = (\psi, Av) \quad \text{———} \quad (7)$$

if A is symmetric

Consider $(A\psi, v)$

$$(A\psi, v) = \int_0^L \left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi}{dx^2}\right) + \frac{d}{dx}\left(c(x)\frac{d\psi}{dx}\right) + d(x)\psi\right) v\, dx$$

$$= \int_0^L \frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi}{dx^2}\right) v\, dx + \int_0^L \frac{d}{dx}\left(c(x)\frac{d\psi}{dx}\right) v\, dx +$$
$$\int_0^L d(x)\psi v\, dx \quad \text{———} \quad (8)$$

Consider the first term on the right side of (8).

$$\int_0^L \frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi}{dx^2}\right) v\,dx \underset{(\text{IBP twice})}{=} \int_0^L \left(\frac{d^2 v}{dx^2} b(x)\right)\frac{d^2\psi}{dx^2}\,dx + \left(\frac{d}{dx}\left(b(x)\frac{d^2\psi}{dx^2}\right)v\right)\Bigg|_0^L$$

$$- \left(b(x)\frac{d^2\psi}{dx^2}\frac{dv}{dx}\right)\Bigg|_0^L \quad \text{——} \ (9)$$

Transfer two orders of differentiation from ψ to $\frac{d^2 v}{dx^2} b(x)$ in the first term on right side of (9).

$$\therefore \int_0^L \frac{d^2}{dx^2}\left(b(x)\frac{d^2\psi}{dx^2}\right) v\,dx = \int_0^L \frac{d^2}{dx^2}\left(b(x)\frac{d^2 v}{dx^2}\right)\psi\,dx + \left(\frac{d^2 v}{dx^2} b(x)\frac{d\psi}{dx}\right)\Bigg|_0^L$$

$$- \left(\frac{d}{dx}\left(b(x)\frac{d^2 v}{dx^2}\right)\psi\right)\Bigg|_0^L + \left(\frac{d}{dx}\left(b(x)\frac{d^2\psi}{dx^2}\right)v\right)\Bigg|_0^L$$

$$- \left(b(x)\frac{d^2\psi}{dx^2}\frac{dv}{dx}\right)\Bigg|_0^L \quad \text{——} \ (10)$$

Consider second term on right side of (8)

$$\int_0^L \frac{d}{dx}\left(c(x)\frac{d\psi}{dx}\right) v\,dx \underset{(\text{IBP twice})}{=} \int_0^L \frac{d}{dx}\left(c(x)\frac{dv}{dx}\right)\psi\,dx$$

$$+ \left(c(x)\frac{d\psi}{dx} v\right)\Bigg|_0^L - \left(c(x)\frac{dv}{dx}\psi\right)\Bigg|_0^L$$

using (10) and (11) in (8) (11)

$$\left(A\psi, v\right) = \int_0^L \left(\frac{d^2}{dx^2}\left(b(x)\frac{d^2 v}{dx^2}\right) - \frac{d}{dx}\left(c(x)\frac{dv}{dx}\right) + \alpha v\right)\psi\,dx$$

$$+ \langle A\psi, v\rangle_M. \quad \text{——} \ (12)$$

$$(A\psi, v) = (\psi, A^* v) + \langle A\psi, v\rangle_\Gamma \quad \text{—— (13)}$$

in which

$$A^* = \frac{d^2}{dx^2}\left(b(x)\frac{d^2}{dx^2}\right) + \frac{d}{dx}\left(c(x)\frac{d}{dx}\right) + d(x) = A$$

Simplification of Concomitant $\langle A\psi, v\rangle_\Gamma$

$$\langle A\psi, v\rangle_\Gamma = \left(\frac{d^2 v}{dx^2}\, b(x)\frac{d\psi}{dx}\right)\Big|_0^L - \left(\frac{d}{dx}\left(b(x)\frac{d^2 v}{dx^2}\right)\psi\right)\Big|_0^L$$
$$+ \left(\frac{d}{dx}\left(b(x)\frac{d^2\psi}{dx^2}\right)v\right)\Big|_0^L - \left(\left(b(x)\frac{d^2\psi}{dx^2}\right)\frac{dv}{dx}\right)\Big|_0^L$$
$$+ \left(c(x)\frac{d\psi}{dx}v\right)\Big|_0^L - \left(c(x)\frac{dv}{dx}\psi\right)\Big|_0^L \quad \text{—— (14)}$$

Recall BCs in (2)..

$$\left.\begin{aligned}
\psi(0) &= \psi_0 && \Rightarrow \quad v(0) = 0\\
\psi(L) &= \psi_L && \Rightarrow \quad v(L) = 0\\
\left(b(x)\frac{d^2\psi}{dx^2}\right)\Big|_{x=0} &= M_0 && \Rightarrow \quad b(0)\frac{d^2 v}{dx^2}\Big|_0 = 0\\
\left(b(x)\frac{d^2\psi}{dx^2}\right)\Big|_{x=L} &= M_L && \Rightarrow \quad b(L)\frac{d^2 v}{dx^2}\Big|_L = 0
\end{aligned}\right\} \quad \text{—— (15)}$$

we expand boundary terms in (14) and then use (15)

$$\begin{aligned}
\langle A\psi, v\rangle_M = & \left(\underbrace{\frac{d^2v}{dx^2}}_{0} b(x)\frac{d\psi}{dx}\right)\Big|_L - \left(\underbrace{\frac{d^2v}{dx^2}}_{0} b(x)\frac{d\psi}{dx}\right)\Big|_0 \\
& - \left(\frac{d}{dx}\left(b(x)\frac{d^2v}{dx^2}\right)\underbrace{\psi}_{\psi_L}\right)\Big|_L + \left(\frac{d}{dx}\left(b(x)\frac{d^2v}{dx^2}\right)\underbrace{\psi}_{\psi_0}\right)\Big|_0 \\
& + \left(\frac{d}{dx}\left(b(x)\frac{d^2\psi}{dx^2}\right)\underbrace{v}_{0}\right)\Big|_L - \left(\frac{d}{dx}\left(b(x)\frac{d^2\psi}{dx^2}\right)\underbrace{v}_{0}\right)\Big|_0 \\
& - \left(\underbrace{b(x)\frac{d^2\psi}{dx^2}}_{M_L}\frac{dv}{dx}\right)\Big|_L + \left(\underbrace{b(x)\frac{d^2\psi}{dx^2}}_{M_0}\frac{dv}{dx}\right)\Big|_0 \\
& \left(c(x)\frac{d\psi}{dx}\underbrace{v}_{0}\right)\Big|_L - \left(c(x)\frac{d\psi}{dx}\underbrace{v}_{0}\right)\Big|_0 \\
& - \left(c(x)\frac{dv}{dx}\underbrace{\psi}_{\psi_L}\right)\Big|_L + \left(c(x)\frac{dv}{dx}\underbrace{\psi}_{\psi_0}\right)\Big|_0 \qquad \text{——(16)}
\end{aligned}$$

$\therefore$

$$\begin{aligned}
\langle A\psi, v\rangle_M = & -\left(\frac{d}{dx}\left(b(x)\frac{d^2v}{dx^2}\right)\right)\Big|_L \psi_L + \left(\frac{d}{dx}\left(b(x)\frac{d^2v}{dx^2}\right)\right)\Big|_0 \psi_0 \\
& - \left(\frac{dv}{dx}\right)\Big|_L M_L + \left(\frac{dv}{dx}\right)\Big|_0 M_0 \\
& - \left(c(x)\frac{dv}{dx}\right)\Big|_L \psi_L + \left(c(x)\frac{dv}{dx}\right)\Big|_0 \psi_0 \qquad \text{——(17)}
\end{aligned}$$

This is the final simplified form of the Concomitant $\langle A\psi, v\rangle_M$.

1. $\langle A\psi, v\rangle_\Gamma$ goes to zero if $\psi_0 = \psi_L = M_0 = M_L = 0$ i.e if the boundary conditions are homogeneous otherwise it is nonzero.

2. $A^* = A$ i.e the adjoint of the operator is same as the operator

3. Based on 1 and 2 The differential operator A is not symmetric with the BCs in (2)

4. If all BCs are homogeneous, then A is symmetric

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Prob. 2.11

$$-\frac{d}{dx}\left(\phi \frac{d\phi}{dx}\right) + f(x) = 0 \quad \forall\, x \in \Omega = (0,1) \subset R^1 \qquad (1)$$

BCs: $\left.\frac{d\phi}{dx}\right|_{x=0} = q_0$ and $\phi(1) = \sqrt{2}$ —— (2)

(a) Differential operator A

$$A = -\frac{d}{dx}\left(\phi \frac{d}{dx}\right) \qquad \text{——— (3)}$$

$\therefore \quad A\phi - f = 0 \quad \forall\, x \in (0,1) \subset R^1$ —— (4)

(b) Linearity of A

Let $\phi_1, \phi_2 \in D_A$

and $\alpha_1, \alpha_2 \in R$

Then $A(\alpha_1\phi_1 + \alpha_2\phi_2) = \alpha_1 A\phi_1 + \alpha_2 A\phi_2$ must hold —— (5)

if the operator A is linear

Consider $A(\alpha_1\phi_1 + \alpha_2\phi_2) = -\frac{d}{dx}\left((\alpha_1\phi_1 + \alpha_2\phi_2)\frac{d}{dx}(\alpha_1\phi_1 + \alpha_2\phi_2)\right)$ (6)

and

$$\alpha_1 A\phi_1 = -\alpha_1 \frac{d}{dx}\left(\phi_1 \frac{d\phi_1}{dx}\right) \quad \text{—— (7)}$$

$$\alpha_2 A\phi_2 = -\alpha_2 \frac{d}{dx}\left(\phi_2 \frac{d\phi_2}{dx}\right) \quad \text{—— (8)}$$

Clearly sum of (7) and (8) is not equal to (6), hence for this operator A (5) does not hold.

Therefore the differential operator A in (3) is not linear.

- Since the differential operator is not linear it cannot be symmetric.
- In this case A^* is not meaningful.
