Differential Equations for Engineers: the Essentials

Class 2 notes

Agenda: Class 2

First order linear differential equations:

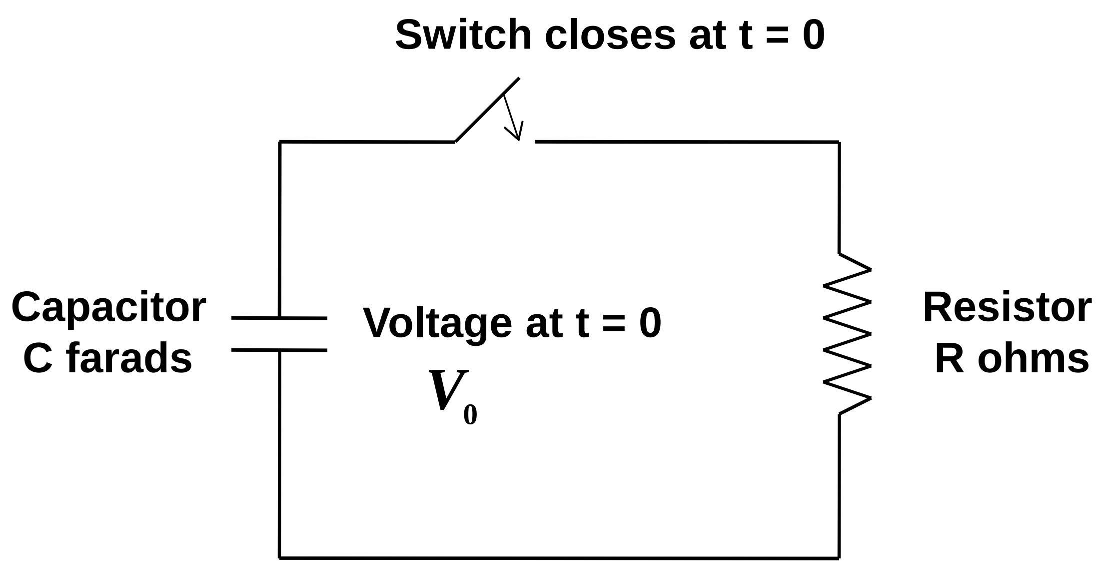
(1) Engineering example: RC circuit
(2) General solution of homogeneous equation
(3) In-class homogeneous problems
(4) Example: Exposed water pipe in cyclical air temperature
(5) General solution of nonhomogeneous equation
(6) In-class nonhomogeneous problems

Homework Assignment 2

First Order Linear Differential Equations

Example: The RC Electrical Circuit

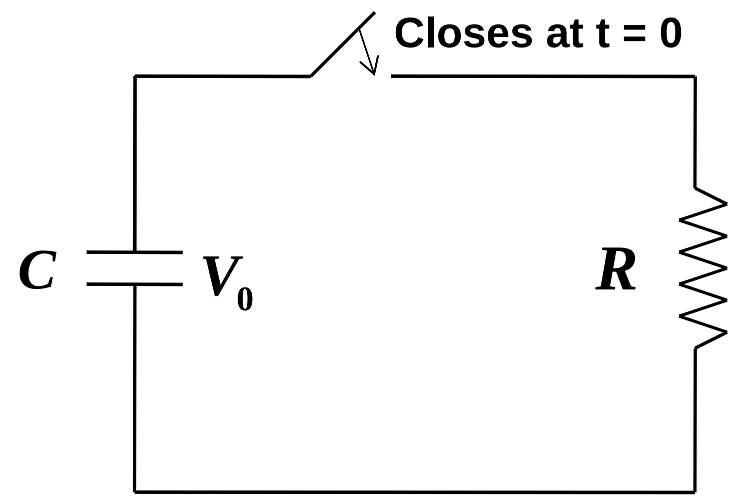
Example: RC Electrical Circuit



Example: RC Circuit (2)

Kirchhoff's Law:

Sum of voltage drops around a closed circuit = 0



Voltage drop over a capacitor:

$$V = q/C \quad q = \text{charge on capacitor}$$

Voltage drop over a resistor:

$$V = IR \qquad I = \text{current through resistor}$$

Conservation of electrical charge:

$$\frac{dq}{dt} = I$$

Example: RC Circuit (3)

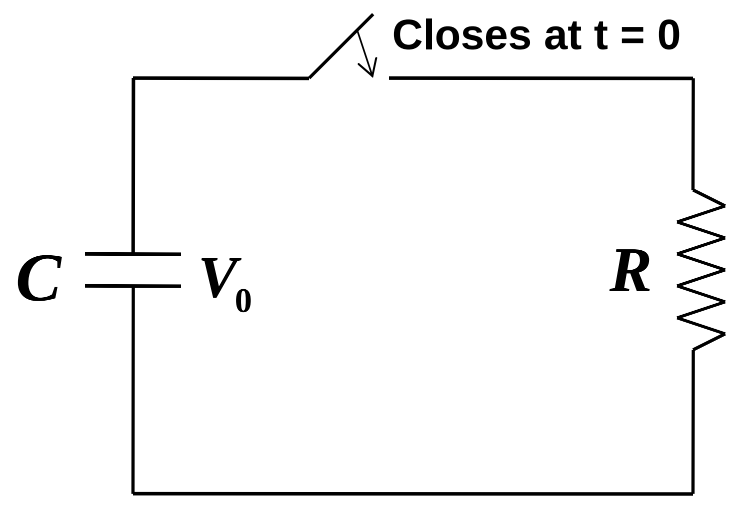
Resulting differential equation

$$IR + V = 0$$

$$\frac{dq}{dt} R + V = 0$$

$$RC \frac{dV}{dt} + V = 0$$

$$\frac{dV}{dt} + \frac{1}{RC} V = 0$$



Initial condition:

$$V(0) = V_0$$

Example: RC Circuit (4)

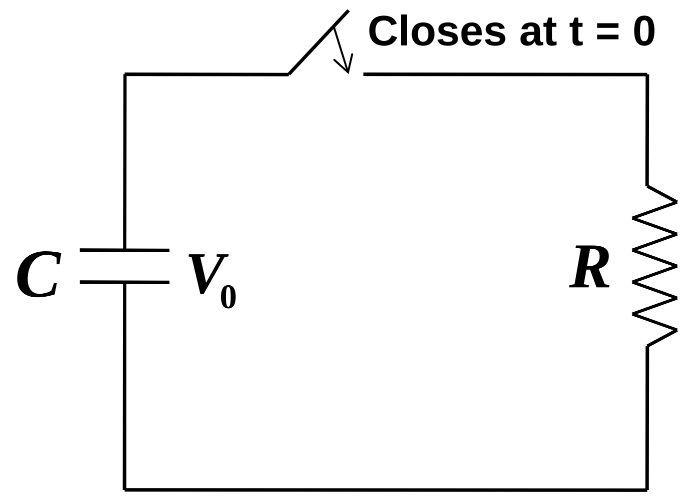
This is a linear first order ODE. To solve it, we separate variables: i.e., we put all terms involving *V* on the left side and all terms involving *t* on the right: specifically, we divide by *V,* move *1/RC* to the right side and multiply by *dt* :

$$\frac{dV}{dt} + \frac{1}{RC}V = 0$$

$$\frac{1}{V}\frac{dV}{dt} + \frac{1}{RC} = 0$$

$$\frac{1}{V}\frac{dV}{dt} = -\frac{1}{RC}$$

$$\frac{dV}{V} = -\frac{1}{RC}dt$$



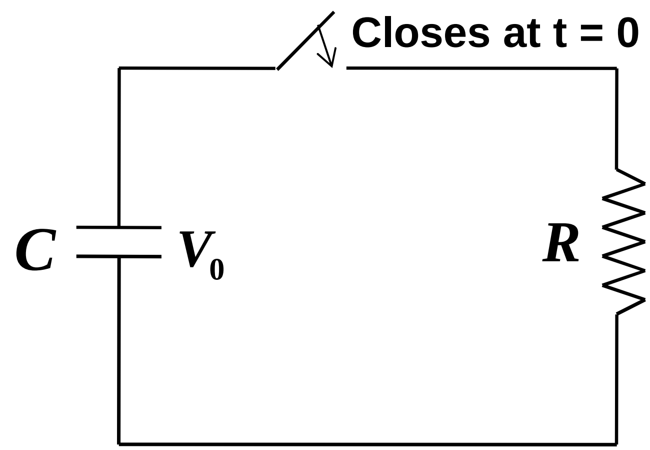
The text justifies this short-cut procedure

Example: RC Circuit (5)

Next, we integrate both sides:

$$\int_{V(0)}^{V(t)} \frac{dV}{V} = -\int_0^t \frac{dt}{RC}$$

$$\ln\left(\frac{V(t)}{V(0)}\right) = -t/RC$$



Taking the exponential of both sides:

$$\exp\left(\ln\left(\frac{V(t)}{V(0)}\right)\right) = \exp(-t/RC)$$

$$\frac{V(t)}{V(0)} = e^{-t/RC}$$

Example: RC Circuit (6)

Hence:

$$V(t) = V(0)e^{-t/RC}$$

We require that the voltage over the capacitor at time 0 be given by

$$V(0) = V_0$$

and so

$$V(t) = V_0 e^{-t/RC}$$

Example: RC Circuit (7)

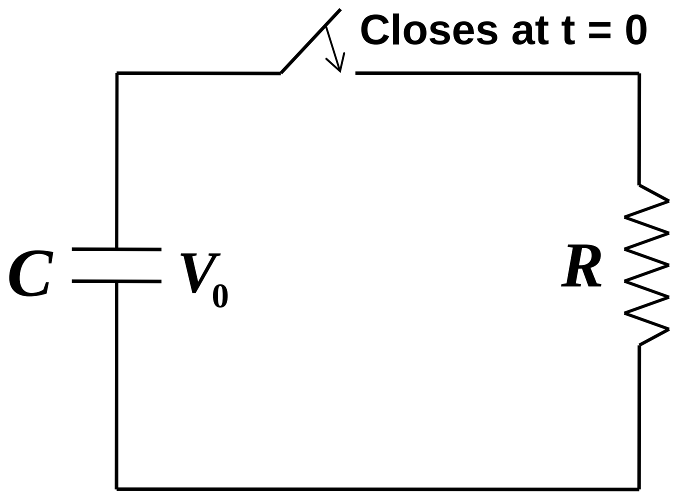
In summary, the solution to

$$\frac{dV}{dt} + \frac{1}{RC} V = 0$$

$$V(0) = V_0$$

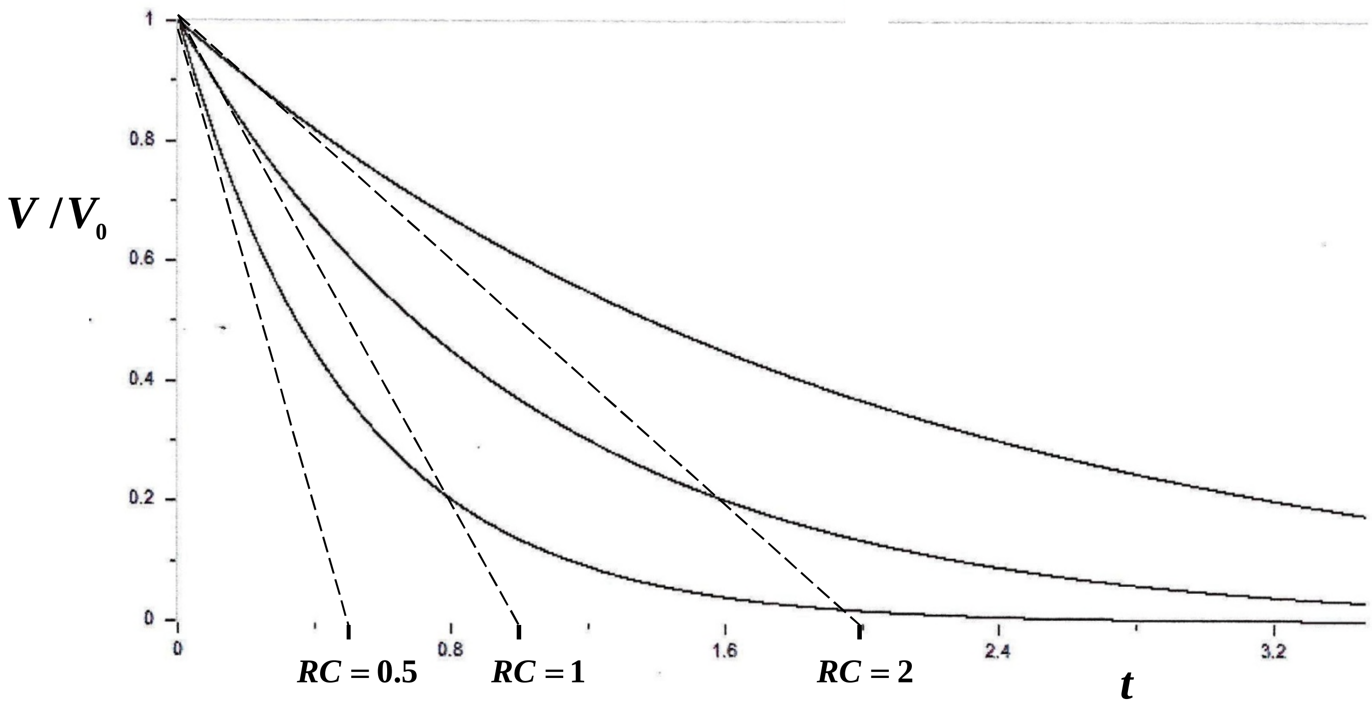
is

$$V(t) = V_0 e^{-t/RC}$$



The product RC has the dimension of time and is called the time constant for the circuit

Example: RC Circuit (8)



First Order Linear Differential Equations

General Solution of Homogeneous Equation

General Solution of Homogeneous Equation

The general form of a first order linear time-varying ordinary differential equation (ODE) is

$$\frac{dy}{dt} + p(t)\,y = 0 \qquad\qquad y(t_0) = y_0 \qquad\qquad \textbf{Equation 1}$$

How do we solve it?

General Solution of Homogeneous Equation (2)

We separate variables

$$\frac{dy}{y} = -p(t)dt$$

Integrate both sides

$$\int_{y_0}^{y(t)} \frac{dy}{y} = -\int_{t_0}^{t} p(\tau)d\tau$$

$$\ln\left(\frac{y(t)}{y_0}\right) = -\int_{t_0}^{t} p(\tau)d\tau$$

Take the exponential of both sides

$$\exp\left(\ln\left(\frac{y(t)}{y_0}\right)\right) = \exp(-\int_{t_0}^{t} p(\tau)d\tau)$$

$$\frac{y(t)}{y_0} = \exp(-\int_{t_0}^{t} p(\tau)d\tau)$$

General Solution of Homogeneous Equation (3)

Summary:

The solution of

$$\frac{dy}{dt} + p(t)y = 0 \qquad\qquad y(t_0) = y_0$$

is

$$y(t) = y_0 \exp\left(-\int_{t_0}^{t} p(\tau)d\tau\right)$$

Success depends entirely on being able to do the integral

Homogeneous First Order Linear ODEs: In-class problems

$$\frac{dy}{dt} + ky = 0$$
$$y(0) = a$$

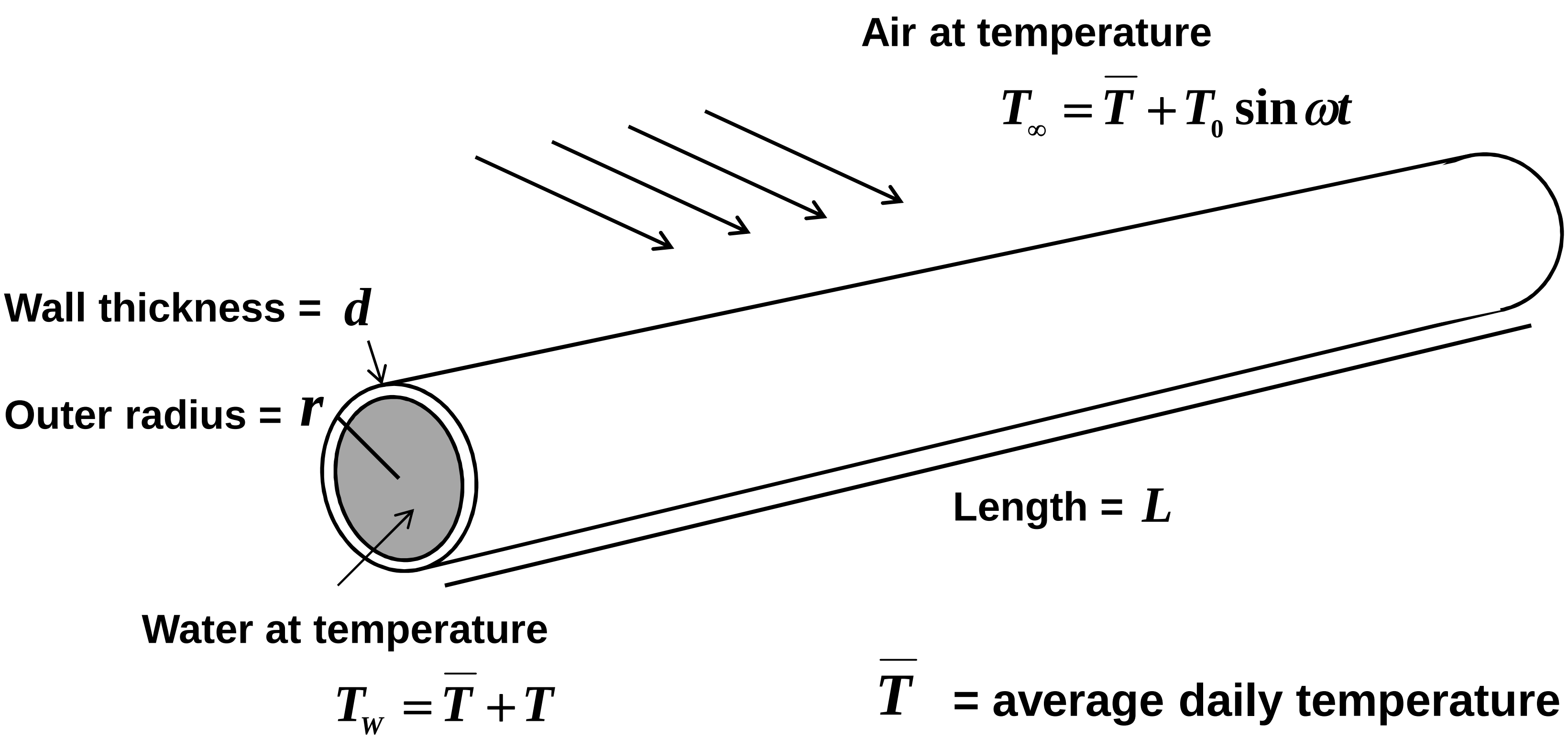
$$\frac{dy}{dt} + ty = 0$$
$$y(0) = b$$

$$\frac{dy}{dt} + \left(\frac{1}{t}\right)y = 0$$
$$y(1) = c$$

First Order Linear Differential Equations

Example: Exposed water pipe in cyclical ambient temperature

Exposed Water Pipe in Cyclical Ambient Temperature



$\overline{T}$ = average daily temperature

Exposed Water Pipe in Cyclical Ambient Temperature (2)

Assuming pipe wall is thin and made of material that is a good heat conductor, by Newton's law of cooling, the heat transferred from air to water is

$$q = hA(T_\infty - T_W)$$

where

$A =$ **Exposed surface area of the pipe**

$h =$ **Convection coefficient**

Exposed Water Pipe in Cyclical Ambient Temperature (3)

The thermal energy stored in the water is

$$E = mcT_W$$

where

$m =$ **mass of the water**

$c =$ **specific heat of water**

Exposed Water Pipe in Cyclical Ambient Temperature (4)

Key physical principle:

$$\frac{dE}{dt} = q$$

which leads to

$$mc\frac{d}{dt}\left(\bar{T} + T\right) = hA\left((\bar{T} + T_0 \sin \omega t) - (\bar{T} + T)\right)$$

$$mc\frac{dT}{dt} + (hA)T = (hA)T_0 \sin \omega t$$

$$\frac{dT}{dt} + \lambda T = \lambda T_0 \sin \omega t$$ **where** $$\lambda = \frac{hA}{mc}$$

Exposed Water Pipe in Cyclical Ambient Temperature (5)

How do we solve

$$\frac{dT}{dt} + \lambda T = \lambda T_0 \sin \omega t \qquad ? \qquad \text{Equation 2}$$

Let's be more inclusive and ask how do we solve the general linear first order nonhomogeneous equation

$$\frac{dy}{dt} + p(t)y = g(t) \qquad \text{Equation 3}$$

$$y(t_0) = y_0$$

General Solution to Nonhomogeneous Linear First Order ODEs

We begin by searching for an integrating factor $\mu(t)$ that, when multiplied into the equation, turns the left-hand side into

$$\frac{d}{dt}(\mu(t)y)$$

Multiplying Equation 3 by $\mu(t)$:

$$\mu(t)\frac{dy}{dt}+\mu(t)p(t)y=\mu(t)g(t)$$

Search for $\mu(t)$ such that left hand side is

$$\mu(t)\frac{dy}{dt}+\mu(t)p(t)y=\frac{d}{dt}(\mu(t)y)$$

General Solution to Nonhomogeneous Linear First Order ODEs (2)

We must have

$$\mu(t)\frac{dy}{dt}+\mu(t)p(t)y=\frac{d}{dt}(\mu(t)y)=\mu(t)\frac{dy}{dt}+\frac{d\mu}{dt}y$$

which means that

$$\frac{d\mu}{dt}=p(t)\mu(t)$$

$$\frac{1}{\mu(t)}\frac{d\mu}{dt}=p(t)$$

General Solution to Nonhomogeneous Linear First Order ODEs (3)

$$\frac{d}{dt}(\ln \mu(t)) = p(t)$$

$$\ln(\mu(t)/\mu(t_0)) = \int_{t_0}^{t} p(u)du$$

$$\mu(t) = \mu(t_0)\exp\int_{t_0}^{t} p(u)du \qquad \text{Equation 4}$$

This is the desired integrating factor.

But we can simplify the form.

General Solution to Nonhomogeneous Linear First Order ODEs (4)

We do not know what value to assign to $\mu(t_0)$ but it turns out not to matter. (The value cancels out.) So we set

$$\mu(t_0) = 1$$

It also suffices to use the indefinite integral form:

$$\mu(t) = \exp \int^t p(u)du \qquad \textbf{Equation 5}$$

You should remember, or be able to derive, Equation 5

Exposed Water Pipe in Cyclical Ambient Temperature (6)

For the water pipe temperature problem (Equation 2):

$$\frac{dy}{dt} + p(t)y = g(t)$$

becomes

$$\frac{dT}{dt} + \lambda T = \lambda T_0 \sin \omega t$$

so

$$p(t) = \lambda$$

$$\mu(t) = \exp\left(\int^t p(u)du\right) = \exp\left(\int^t \lambda du\right) = e^{\lambda t}$$

Exposed Water Pipe in Cyclical Ambient Temperature (7)

Applying the integration factor to Equation 2:

$$e^{\lambda t}\left(\frac{dT}{dt}+\lambda T\right)=e^{\lambda t}(\lambda T_0 \sin\omega t)$$

$$\frac{d}{dt}\left(e^{\lambda t}T\right)=\lambda T_0 e^{\lambda t}\sin\omega t$$

Now the value of the integration factor becomes clear:
We can solve the problem with an integration:

$$e^{\lambda t}T(t)-T(0)=\lambda T_0\int_0^t e^{\lambda\tau}\sin\omega\tau d\tau$$

$$T(t)=T(0)e^{-\lambda t}+e^{-\lambda t}\int_0^t e^{\lambda\tau}\sin\omega\tau d\tau$$

Exposed Water Pipe in Cyclical Ambient Temperature (9)

After performing the integral we have

$$T(t) = T_I e^{-\lambda t} + \left(\frac{\lambda}{\lambda^2 + \omega^2} \right) T_0 (\lambda \sin(\omega t) - \omega \cos(\omega t) + \omega e^{-\lambda t})$$

where

$$T_I = T(0) = T_W(0) - \overline{T}$$

Inhomogeneous First Order Linear ODEs: In-class Problems

$$\frac{dy}{dt} + \left(\frac{2}{t}\right) y = 4$$

$$y(1) = 2$$

$$\frac{dy}{dt} + 4\left(\frac{e^{4t} - e^{-4t}}{e^{4t} + e^{-4t}}\right) y = e^{3t}$$

$$y(0) = 6$$

$$\frac{dy}{dt} - (\tan t) y = \sec t$$

$$y(0) = 0$$

Homework Assignment 2

<u>In text</u>:

Read: Chapter 2

Work: On course website: Homework Assignment #2 Problems

Solutions for Homework Assignment #2 Problems will be provided on course website on (date)

Always read over the day's lecture notes and be sure you understand them.