Differential Equations for Engineers: the Essentials

Supplement to Class 4 Notes

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Differences Between Linear and Nonlinear ODEs in Their Input / Output Response

Input / Output Characteristics for a Linear System

In a linear system of any order (time-varying or time-invariant):

If the output is $y_1(t)$ **when the input is** $u_1(t)$ **and the output is** $y_2(t)$ **when the input is** $u_2(t)$
then the output is $ay_1(t)+by_2(t)$ **when the input is** $au_1(t)+bu_2(t)$ **for any constants** a **and** b

In a linear <u>time-invariant</u> system of any order, after transients have died away, when the input is a sine-wave of a given frequency the output is a steady-state oscillation of only that frequency.

These characteristics are <u>not</u> generally true of nonlinear systems

Existence and Uniqueness of Solutions to Linear ODEs

Theorem: **Given a linear nth order ODE:**

$$\frac{d^n y}{dt^n} + a_1(t)\frac{d^n y}{dt^n} + \ldots + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = g(t)$$

with initial conditions

$$\frac{d^{n-1} y}{dt^{n-1}}(0) = y_0^{(n-1)} \qquad \frac{d^{n-2} y}{dt^{n-2}}(0) = y_0^{(n-2)} \qquad \ldots \qquad \frac{dy}{dt}(0) = y_0' \qquad y(0) = y_0$$

If the coefficients $a_i(t)$ and the input $g(t)$ are continuous for all t then there exists a unique solution to the ODE satisfying the initial conditions for all t .

(Stated without proof.)

This is not generally true of nonlinear ODEs.

Key Points from the Following Nonlinear Example

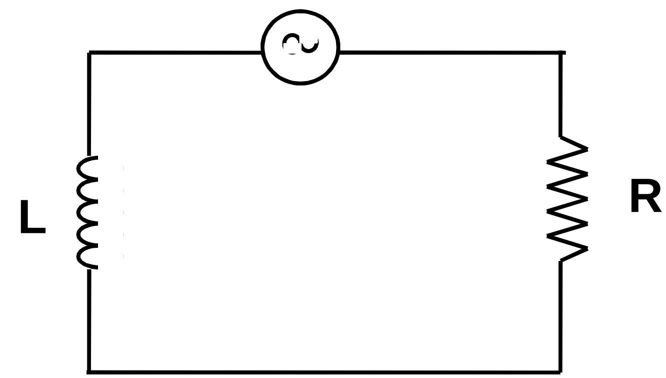
The method of <u>successive approximations</u> (solving a sequence of linear equations to approximate the solution of a nonlinear equation) is a powerful tool.

Nonlinearities in systems designed to be linear cause distortions in the frequency response, introducing “harmonics” (oscillations that are multiples of the input frequency).

Nonlinear Example: LR Circuit

Kirchhoff's Law:

Sum of voltage drops around a closed circuit = 0



Voltage drop over an inductor:

$$V = L\frac{dI}{dt}$$

Voltage drop over a resistor:

$$V = IR$$

Voltage drop over source:

$$V = -V_0 \sin(\omega t)$$

Resulting equation :

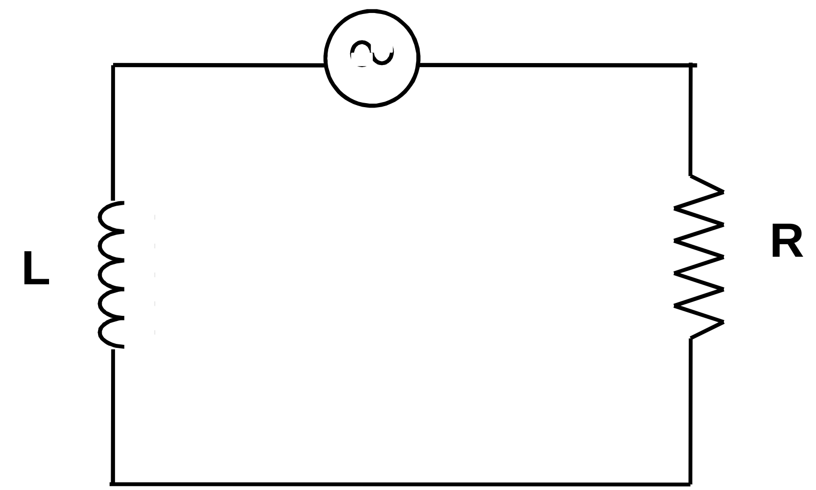
$$L\frac{dI}{dt} + IR = V_0 \sin(\omega t)$$

Nonlinear Example: LR Circuit (2)

The solution to

$$L\frac{dI}{dt} + IR = V_0 \sin(\omega t)$$

$$I(0) = 0$$



is

$$I(t) = \frac{\lambda}{\sqrt{\lambda^2+\omega^2}}(V_0/R)\sin(\omega t - \theta_1) + \frac{\lambda\omega}{\lambda^2+\omega^2}(V_0/R)e^{-\lambda t}$$

where

$$\theta_1 = \arctan(\omega/\lambda) \qquad \lambda = R/L$$

Nonlinear Example: Recalling the LR Circuit (3)

Input / output response:

The "input" to the "system" is the voltage source. The "output" is the current (or voltage) over the resistor.

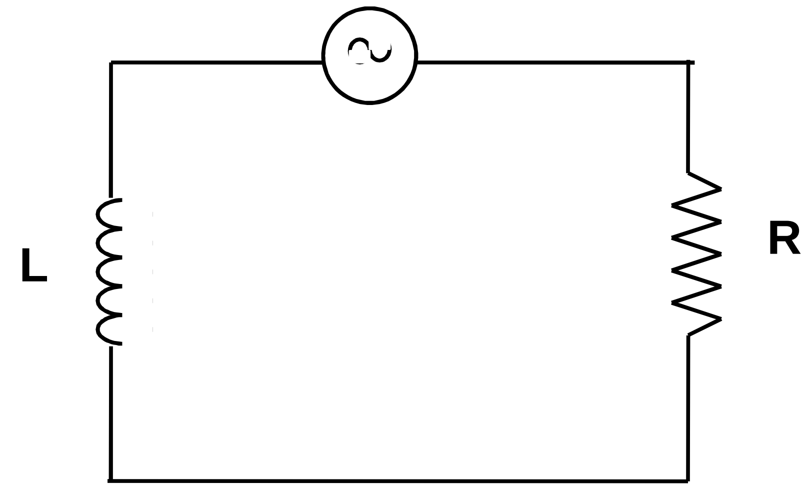
Ignoring the transient, the system passes the sine wave frequency perfectly - it introduces no other frequencies.

We only discuss frequency response in the context of time-invariant systems.

What if the Resistor is Slightly Nonlinear?

Instead of

$$L\frac{dI}{dt} + IR = V_0 \sin(\omega t)$$



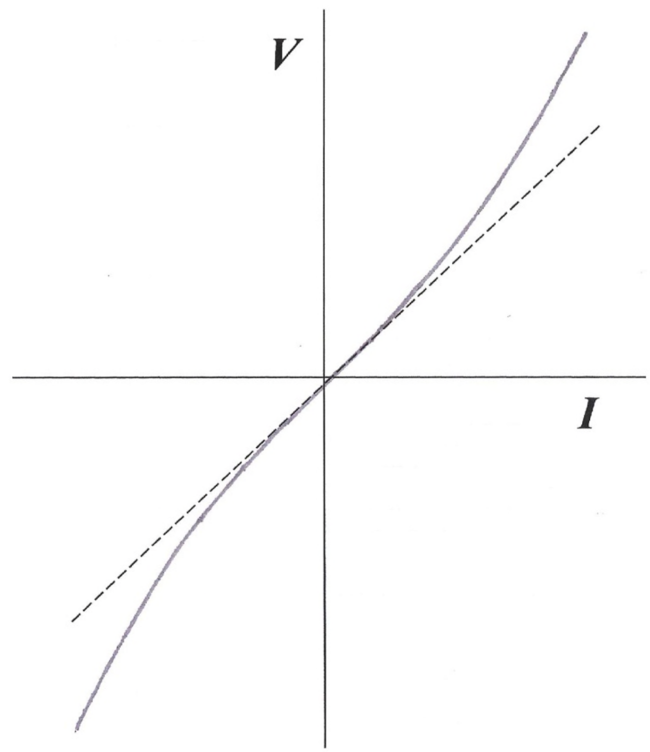
suppose we have

$$L\frac{dI}{dt} + R(I + \varepsilon I^3) = V_0 \sin \omega t$$

$$\frac{dI}{dt} + \lambda(I + \varepsilon I^3) = \lambda\left(\frac{V_0}{R}\right)\sin(\omega t) = \lambda I_0 \sin(\omega t)$$

Equation 1

What if the Resistor is Slightly Nonlinear? (2)



Nonlinear resistor characteristic

Approach to Solution of Slightly Nonlinear ODE

Since ε is small, consider the successive approximations

$$\frac{dI_1}{dt} + \lambda I_1 = \lambda I_0 \sin(\omega t)$$

$$I_1(0) = 0$$

Equation 2

$$\frac{dI_2}{dt} + \lambda I_2 = \lambda I_0 \sin(\omega t) - \varepsilon\lambda I_1^3(t)$$

$$I_2(0) = 0$$

Equation 3

$$\frac{dI_3}{dt} + \lambda I_3 = \lambda I_0 \sin(\omega t) - \varepsilon\lambda I_2^3(t)$$

$$I_3(0) = 0$$

Equation 4

Approach to Solution of Slightly Nonlinear ODE (2)

We are solving the <u>nonlinear</u> Equation 1 approximately by the sequential solution of a series of <u>linear</u> Equations 2, 3 and 4.

The general solution of

$$\frac{dI}{dt} + \lambda I = g(t) \qquad \text{is} \qquad I(t) = \int_0^t e^{-\lambda(t-\tau)} g(\tau)\, d\tau \qquad \text{Equation 5}$$

$$I(0) = 0$$

Using Equation 5, we have already found that the solution to Equation (2) is

$$I_1(t) = \frac{\lambda}{\sqrt{\lambda^2 + \omega^2}} I_0 \sin(\omega t - \theta_1) + \frac{\lambda\omega}{\lambda^2 + \omega^2} I_0 e^{-\lambda t}$$

In what follows we will ignore the transient. We are examining the steady state frequency response.

Approach to Solution of Slightly Nonlinear ODE (3)

Repeating Equation 3:

$$\frac{dI_2}{dt} + \lambda I_2 = \lambda I_0 \sin\omega t - \lambda\varepsilon I_1^3$$

This has solution

$$I_2(t) = \int_0^t e^{-\lambda(t-\tau)}\left(\lambda I_0 \sin(\omega\tau) - \lambda\varepsilon I_1^3\right)d\tau$$

Now

$$\int_0^t e^{-\lambda(t-\tau)}\lambda I_0 \sin(\omega\tau)d\tau = I_1(t)$$

so

$$I_2(t) = I_1(t) - \lambda\varepsilon\int_0^t e^{-\lambda(t-\tau)} I_1^3(\tau)d\tau$$

Approach to Solution of Slightly Nonlinear ODE (4)

Continuing in this way, we find

$$I_{n+1}(t) = I_1(t) - \lambda\varepsilon\int_0^t e^{-\lambda(t-\tau)} I_n^3(\tau)d\tau$$

which one can show can be written as

$$I_n(t) = I_1(t) - G_1 I_0 \eta_n(t)$$

where

$$G_1 = \frac{\lambda}{\sqrt{\lambda^2+\omega^2}} \qquad I_0 = V_0 / R$$

$$\eta_n(t) = \sum_{m=1}^{N_n} k_{mn} \sin(m\omega t - \phi_{mn})$$

Note the higher frequencies – the harmonics

Approach to Solution of Slightly Nonlinear ODE (4)

***Challenge problem*:**

Given the steady state solution

$$I_1(t) = G_1 I_0 \sin(\omega t - \theta_1)$$

find the steady state component of $I_2(t)$