2.1 Sequences and Convergence

1. For the three sequences given:

	ϵ	1.000	0.100	0.010	0.001
(a)	N	1	$\sqrt{10} \approx 3.16$	$\sqrt{100} = 10$	$\sqrt{1000} \approx 31.6$
(b)	N	1	100	$10,000$	$1,000,000$
(c)	N	1	$e^9 \approx 8103$	e^{99}	e^{999}

2. (a) If a given N works for $\epsilon = 0.001$, then the same N works for larger ϵ. We could use this N in all positions in the row.
   (b) Any N works in this case; we could use $N = 0$ in all positions, for example.
   (c) The number of digits in a number changes after 9, 99, 999, etc. So the second-row entries would be 9, $10^9 - 1$, $10^{99} - 1$, and $10^{999} - 1$.

3. (a) Set $\epsilon = 0.1$ and choose a corresponding N as in the definition. All x_n with $n > N$ lie in the desired interval.
   (b) Set $\epsilon = 0.001$ and choose a corresponding N as in the definition. Then all x_n with $n > N$ lie in $(4.999, 5.001)$, and therefore exceed 4.999.
   (c) If we set $\epsilon = 1$ and choose a corresponding N, then all x_n with $n > N$ lie in $(4, 6)$. Thus it is possible that $x_n > 6$ only for members of the finite set $\{x_1, x_2, \ldots x_N\}$.

4. (a) Yes. The divergent sequence $3, 0, 3, 0, 3, 0, 3, 0, \ldots$ has infinitely many terms with value 3.
   (b) No. If, say, $\{x_n\} = 3$ for all $n > N$, then this value of N works in the definition of convergence for any $\epsilon > 0$.

6. Suppose that $\{x_n\}$ converges to $L = 1000$. Let $\epsilon = 1$ and choose N as in the definition of convergence. Then we have $x_n \in (999, 1001)$ for all $n > N$, which means that 1001 is an upper bound for these x_n. The remaining x_n form a finite set, which is automatically bounded above. This contradicts the assumption of boundedness.

7. (a) Algebraic manipulation gives
$$|a_n - L| = \left| \frac{2n}{3n+5} - \frac{2}{3} \right| = \frac{10}{9n+15}$$
   and
$$\frac{10}{9n+15} < \epsilon \iff \frac{10/\epsilon - 15}{9} < n.$$
   Thus, for given $\epsilon > 0$, the value $N = \frac{10/\epsilon - 15}{9}$ works in the definition. A formal proof resembles that in Example 2.
   (b) The proof is like that for (a), except that now we have
$$|b_n - L| = \frac{899900}{9n+15} < \epsilon \iff \frac{899900/\epsilon - 15}{9} < n.$$
   Thus $N = \frac{899900/\epsilon - 15}{9}$ works for any given $\epsilon > 0$.
   (c) We'll show $c_n \to 0$. Observe first that
$$|c_n - L| = \frac{2n}{3n^2+5} < \frac{2n}{3n^2} = \frac{2}{3n},$$
   and
$$\frac{2}{3n} < \epsilon \iff \frac{2}{3\epsilon} < n.$$
   Here's the formal proof: Let $\epsilon > 0$ be given; set $N = \frac{2}{3\epsilon}$. This N works, since if $n > N$ then
$$|c_n - L| = \frac{2n}{3n^2+5} < \frac{2}{3n} < \frac{2}{3N} = \epsilon.$$

8. Formal proofs resemble those in the preceding problem. Following are sketches.
   (a) The limit is $2/3$. For given $\epsilon > 0$ we can use $N = \frac{15+10/\epsilon}{9}$.
   (b) The limit is $L = 2/3$. Basic algebra gives
$$|b_n - L| = \frac{10 - 2\sin n}{9n + 3\sin n + 15} \le \frac{8}{9n}.$$
   (Convince yourself of the last inequality.) This implies that, for given $\epsilon > 0$ we can use $N = 8/(9\epsilon)$.
   (c) As in the preceding problem, $c_n \to 0$. Indeed,
$$|c_n - 0| = \frac{2n}{3n^2+5} < \frac{2}{3n},$$
   and so, for given $\epsilon > 0$ we can use $N = 2/(3\epsilon)$.

11. (a) We can check explicitly that $1.5^{11} \approx 86.5 < 89 = f_{11}$, and $1.5^{12} \approx 129.7 < 144 = f_{12}$. For the inductive step, note that
$$f_{k+1} > 1.5^{k-1} + 1.5^k = 1.5^{k-1} \cdot 2.5 > 1.5^{k-1} \cdot 1.5^2 = 1.5^{k+1},$$
   as desired. (Note that the proof worked because $1 + 1.5 > 1.5^2$. A similar result can be shown to hold if 1.5 is replaced by any number b with $1 + b > b^2$.)
   (b) Using technology one can check that $n \ge 72$ works.

12. (a) The first few terms are $1/1, 2/1, 3/2, 5/3, 8/5$; the Fibonacci numbers appear as numerators and denominators.
   (b) The claim is obviously true for $n = 1$. The inductive step is easy, too. Clearly, $g_n > 1$ for all n, and so $g_{n+1} = 1+1/g_n < 1 + 1/1 = 2$.
   (c) Easy algebra is enough.

13. (a) For any given $\epsilon > 0$, we see
$$|x_n - 0| = \frac{1}{n} < \epsilon \quad \Longleftrightarrow \quad n > \frac{1}{\epsilon}.$$
   Thus the desired inequality holds for all n except those (finitely many) for which $n \le 1/\epsilon$.
   (b) Show that if $x_n \to 0$ in the sense above, then $x_n \to 0$ in the "official" sense defined in this section. Let $\epsilon > 0$ be given. By hypothesis, $|x_n - 0| \ge \epsilon$ holds for only finitely many n, and we can choose N to the largest such value of n. This N works in the ordinary definition.
   (c) A sequence $\{x_n\}$ does not converge to 0 if, for some $\epsilon > 0$, there are infinitely many x_n with $|x_n| > \epsilon$.
   (d) A sequence $\{x_n\}$ does not converge to 0 if, for some $\epsilon > 0$ and any number N, there is some x_n with $n > N$ and $|x_n| > \epsilon$.

14. The sequence $\{y_n\}$ converges to 212. To see why, notice that
$$|y_n - 212| = |5x_n + 2 - 212| = 5\,|x_n - 42|\,,$$
   and so
$$|y_n - 212| < \epsilon \quad \Longleftrightarrow \quad |x_n - 42| < \frac{\epsilon}{5}.$$
   For a given $\epsilon > 0$, therefore, we can set $\epsilon' = \epsilon/5$, and choose N that works for the sequence $\{x_n\}$ and ϵ'. The same N does what's needed for $\{y_n\}$.

15. Every constant sequence has such a table.

16. A sequence has such a table if and only if the sequence is constant from the sixth term on: $x_6 = x_7 = x_8 = \dots$. (Such a sequence is called *eventually constant.*)

17. If β has decimal expansion $0.d_1d_2d_3d_4d_5\dots$, then the increasing sequence $0.d_1$, $0.d_1d_2$, $0.d_1d_2d_3$, $0.d_1d_2d_3d_4, \dots$ converges to β.

2.2 Working with Sequences

1. Here is a sketch. For any constant $c \neq 0$, we have

$$|ca_n - ca| = |c|\,|a_n - a| < \epsilon \iff |a_n - a| < \frac{\epsilon}{|c|}.$$

Now for any given $\epsilon > 0$, we can choose N so that $|a_n - a| < \epsilon/|c|$ for $n > N$. (Why can we do this?) This N "works" for the given ϵ and the original sequence $\{ca_n\}$. (Assemble the pieces into an efficient proof.)

3. Suppose $x_n \to 0$. To show $|x_n| \to 0$, let $\epsilon > 0$ be given. Because $x_n \to 0$, we can choose N so that $|x_n - 0| = |x_n| < \epsilon$ whenever $n > N$. The same N works for the sequence $\{|x_n|\}$, because if $n > N$, then

$$||x_n| - 0| = ||x_n|| = |x_n| < \epsilon,$$

as desired.

4. (a) The lemma says that there exist both rational and irrational sequences that tend to $\sqrt{2}$, and that these can be chosen to be monotone.

   (b) One possibility is $p_n = \sqrt{2} + 1/n$ for all n.

   (c) One possibility comes from the infinite decimal approximation $\sqrt{2} = 1.414213562\ldots$; we can use $r_1 = 1$, $r_2 = 1.4$, $r_3 = 1.414$, etc. (We need a little fix when a zero-digit comes along.) Another possibility, a bit less concrete, is to use the fact that every interval contains rational numbers.

5. The $n = 1$ (base) case is easy: $s_2 = 1.5 < 2 = s_1$. The inductive step is to show that if $s_n - s_{n-1} < 0$, then $s_{n+1} - s_n < 0$, too. Doing so involves careful but straightforward algebra.

6. For every positive integer n we have the estimate

$$s_n = \frac{1}{1} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}.$$

Clearly, $\sqrt{n} \to \infty$, and so $s_n \to \infty$, too.

7. It is clear that $\{h_n\}$ is increasing, and the inequality $h_{2n} \geq h_n + \frac{1}{2}$ implies that $\{h_n\}$ is unbounded. Since $h_1 = 1$, the inequality means that $h_2 \geq 1.5$, $h_4 \geq 2.0$, $h_8 \geq 2.5$, $h_{2^{1000}} > 501$, etc.

8. (a) Clearly, $T_n < S_n$ for all n, and both sequences appear to converge.

   (b) It is natural (and correct) to guess $T_n = \frac{n}{n+1}$. The inductive proof is straightforward.

   (c) We can use algebraic facts: $T_n = \frac{n}{n+1} = \frac{1}{1+1/n} \to \frac{1}{1+0}$.

   (d) Observe that

$$\frac{1}{n^2+n} \geq \frac{1}{n^2+n^2} = \frac{1}{2n^2}$$

   for all n, and so $S_n \leq 2T_n < 2$ for all $n \geq 1$. Thus $\{T_n\}$ is bounded above, and clearly increasing, so $\{T_n\}$ converges.

9. (a) Let $M > 0$ be given. Since $x_n \to \infty$, we can choose N so $x_n > M$ whenever $n > N$. But then also $-x_n < -M$ when $n > N$. Thus $-x_n \to -\infty$.

   (b) Suppose $x_n \to \infty$. Let $\epsilon > 0$ be given; we need to find N so $n > N$ implies $\left|\frac{1}{x_n}\right| = \frac{1}{x_n} < \epsilon$. Well, if we set $M = 1/\epsilon$ then (since $x_n \to \infty$) we can choose N so $n > N$ implies $x_n > M = 1/\epsilon$. But this implies $1/x_n < \epsilon$, as desired.

   (c) Suppose $x_n < 0$ for all n. Then $x_n \to -\infty$ if and only if $1/x_n \to 0$. This follows from parts (a) and (b): $x_n \to -\infty \iff -x_n \to \infty \iff -\frac{1}{x_n} \to 0 \iff \frac{1}{x_n} \to 0$.

10. The inequality $\left|\sqrt{a_n} - \sqrt{3}\right| < |a_n - 3|$ implies that

$$-|a_n - 3| < \sqrt{a_n} - \sqrt{3} < |a_n - 3|.$$

Since the left and right quantities tend to zero, so must the middle.

Proving the inequality $\left|\sqrt{a_n} - \sqrt{3}\right| < |a_n - 3|$ involves multiplication and division by the "conjugate" expression $\left|\sqrt{a_n} + \sqrt{3}\right|$.

11. (a) The limit is $2/5$.
    (b) The sequence diverges to ∞.
    (c) The sequence converges to $1/2$.
    (d) The sequence diverges to ∞. One strategy is to observe that $\frac{n^2+\arctan n}{n+2} > \frac{n^2-4}{n+2} = n - 2$.

2.3 Subsequences

1. (a) The converse says (falsely) that if $\{x_n\}$ has a convergent subsequence, then $\{x_n\}$ is bounded. Counterexamples are easy to find. The contrapositive says that if $\{x_n\}$ has *no* convergent subsequence, then $\{x_n\}$ is unbounded.
   (b) The BWT says that $\{y_n\}$ has a convergent subsequence. The BWT also applies to $\{z_n\}$, but isn't really needed, since it is easy to see that $z_n \to 0$. Thus, *every* subsequence of $\{z_n\}$ converges to zero.
   (c) If $\{x_n\}$ is bounded, then the BWT guarantees that a convergent subsequence exists. If $\{x_n\}$ is unbounded, then Lemma 2.13 applies. Both can occur, as they do in the sequence $1, 0, 2, 0, 3, 0, dots$.

2. (a) The sequence $1, 2, 3, 1, 2, 3, \ldots$ works.
   (b) The sequence $1, -1, 1/2, -1/2, 1/3, -1/3, \ldots$ works.
   (c) The sequence $0, 1, 0, 2, 0, 3, \ldots$ works.
   (d) The sequence $1, 1, 1/2, 2, 1/3, 3, \ldots$ works.
   (e) The sequence $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots$ works.

3. (a) Given a sequence that lists the rationals, we can just form the subsequence of nonnegative rationals.
   (b) Look at any two successive terms, say p_1 and p_2. Between these two rational numbers like other rationals. If the sequence were monotone, these other rationals would lie between p_1 and p_2 in the sequence.
   (c) We can choose n_1 so that $p_{n_1} = 1$. Then we choose n_2 with $n_2 > n_1$ and $p_{n_2} \geq 2$. Continuing this process completes the proof; details are left to the reader.

4. Note that we can write the given subsequence in the form $x_{n_k} = x_{4241+k}$. Theorem 2.11, page 95, says that if $x_n \to L$, then x_{n_k} converges to L, too. For the converse, suppose that $x_{n_k} \to L$. To show that $x_n \to L$, let $\epsilon > 0$ be given. By hypothesis we can choose N so $n_k > N \implies |x_{n_k} - L| < \epsilon$. Let $N_1 = N + 4241$. This N_1 works; details are left to the reader.

5. (a) Substituting $n = 2k$ gives $x_{2k} = \dfrac{2k}{2n+1}$, which converges to one as $k \to \infty$. Substituting $n = 2k-1$ gives $x_{2k-1} = -\dfrac{2k-1}{2k}$, which converges to -1. With two different subsequence limits, $\{x_n\}$ must diverge.
   (b) Consider any convergent subsequence $x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4}, \ldots$, with limit L. Since this sequence has infinitely many terms, it must include either infinitely many even-indexed or infinitely many odd-indexed terms from the original sequence $\{x_n\}$. If there are, say, infinitely many even-indexed terms, then they form a new subsequence, which obviously tends to one. But every subsequence of a convergent sequence tends to the same limit, so we must have $L = 1$.

6. (a) The sequence $0, 3, 0, 3, 0, 3, \ldots$ is one example.
   (b) The statement $x_n \to 3$ means that, for every $\epsilon > 0$, all but finitely many x_n are within ϵ of 3. Negating this definition means that, for some $\epsilon > 0$, infinitely many x_n are farther than ϵ from 3. These $\{x_n\}$, taken in order, give the desired subsequence.

7. Since $\{x_n\}$ diverges, x_0 is *not* the limit. Negating the definition of convergence to x_0 means that, for some $\epsilon > 0$, *no* N works in the definition of convergence. We'll use this to construct a sequence.

   Because $N = 1$ doesn't work in the definition, there is some term—call it x_{n_1}—for which $|x_{n_1} - x_0| > \epsilon$. Now consider $N = n_1$. But this N doesn't work either, so there must be some term—call it x_{n_2}—for which $n_2 \geq n_1$ and $|x_{n_2} - x_0| > \epsilon$. Continuing this process indefinitely gives the desired subsequence: $n_1 < n_2 < n_3 < \dots$ and $|x_{n_k} - x_0| > \epsilon$ for all k.

8. (a) For $M > 0$, choose N such that $x_n > M$ whenever $n > N$. The same N works for the subsequence $\{x_{n_k}\}$, since if $k > N$, then $n_k \geq k > N$, and so $x_{n_k} > M$, as desired.

   (b) The contrapositive of Theorem 2.11(a) says that if some subsequence $\{x_{n_k}\}$ does *not* converge to L, then the original sequence $\{x_n\}$ doesn't converge to L either. This implies Theorem 2.11(c), because if L is *any* number, then at least one subsequence fails to converge to L, and so L can't be the limit.

9. If $x_n \to x_0$, then we know that every subsequence, and hence every monotone subsequence, must also converge to x_0.

   The other implication is that if every monotone subsequence converges to x_0, then $x_n \to x_0$, too. We prove this by contradiction.

   If x_n does *not* converge to x_0, then for some $\epsilon > 0$ there is a subsequence $\{x_{n_k}\}$ with $|x_{n_k} - x_0| > \epsilon$ for all k. By Proposition 2.12, there is a monotone subsequence $\{y_{n_k}\}$ of $\{x_{n_k}\}$. Now $|x_{n_k} - x_0| > \epsilon$ holds for all the x_{n_k}, and so it certainly holds for all the y_{n_k}, too. In particular, $y_{n_k} \to x_0$ is impossible, which contradicts our assumption about monotone sequences.

10. Since $x_n \to x_0$ there are infinitely many n with $|x_n - x_0| < 1/10$. Choose any of these n to be n_1. Similarly, there are infinitely many n with $|x_n - x_0| < 1/10^2$. Among these n, choose any one with $n > n_1$ to be n_2. Similarly, we can choose $n_3 > n_2$ such that $|x_{n_3} - x_0| < 1/10^3$, and so on.

11. Both parts follow from unraveling the ϵ–N definitions.

2.4 Cauchy Sequences

1. (a) For given $\epsilon > 0$ we can choose $N = 2/\epsilon$. This N works because if $m, n > N$ then $|x_n - x_m| \leq |x_n| + |x_m| = \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

   (b) The sequence is not Cauchy. If we choose, say $\epsilon = 0.001$, then no N works: if N is any positive number, no matter how big, then with $m = N + 1$ and $n = N + 2$ we have $|y_n - y_m| = 2/1234 > 0.001$.

   (c) The sequence is Cauchy. Note that if $n > m$, we have

   $$\begin{aligned} |z_n - z_m| = \left|\frac{n}{n+1} - \frac{m}{m+1}\right| &= \frac{n-m}{(n+1)(m+1)} \\ &< \frac{n-m}{nm} < \frac{n}{nm} = \frac{1}{m}. \end{aligned}$$

   It follows that, for $\epsilon > 0$, we can choose $N = 1/\epsilon$.

   (d) The sequence is Cauchy. If $n > m$, we have

   $$\begin{aligned} |w_n - w_m| = \left|\frac{\sin n}{n^2+1} - \frac{\sin m}{m^2+1}\right| &\leq \left|\frac{\sin n}{n^2+1}\right| + \left|\frac{\sin m}{m^2+1}\right| \\ &\leq \frac{2}{m^2+1} < \frac{2}{m^2}, \end{aligned}$$

   and $2/m^2 < \epsilon$ if $m > \sqrt{2/\epsilon}$. Thus, $N = \sqrt{2/\epsilon}$ works in the definition.

2. For given $\epsilon > 0$, choose N_1 such that $|x_m - x_n| < \epsilon/2$ when $n > m > N_1$, and choose N_2 such that $|y_m - y_n| < \epsilon/2$ when $n > m > N_2$. The number $N = \max\{N_1, N_2\}$ does what's needed.

3. For $\epsilon > 0$ choose N that works for ϵ in the sense of Definition 2.16. This same N works in the definition of convergence to zero. To see why, let $m > N$ be given. Then choose any n of the form $n = 10^k$, with $n > m$. (This can certainly be done; $n = 10^m$ is one possibility.) Then we have $n > m > N$, and so $|x_m - x_n| = |x_m - 0| < \epsilon$, as desired.

4. Set $\epsilon = 1$ (smaller ϵ works, too) and choose N as in the Cauchy definition. This N works in the problem, since if $n > m \geq N$ then $|x_n - x_m| < 1$, which can occur for *integers* x_n and x_m only if $x_n = x_m$.

5. (a) Any sequence of rationals tending to an irrational will do.

   (b) Yes. The only Cauchy sequences of integers are eventually constant.

   (c) $[0, 1]$ is complete because if $\{x_n\}$ is any Cauchy sequence in $[0, 1]$, then $\{x_n\}$ tends to some limit L. Because $0 \leq x_n \leq 1$ for all n, L lies in $[0, 1]$, too.

   $(0, 1]$ is not complete; $\{1/n\}$ is a Cauchy sequence in $(0, 1]$, but its limit, zero, lies outside $(0, 1]$.

6. (a) For any m and n,

$$|y_n - y_m| = |5x_n + 2 - 5x_m - 2| = 5\,|x_n - x_m|\,.$$

   It follows that

$$|y_n - y_m| < \epsilon \quad \iff \quad |x_n - x_m| < \frac{\epsilon}{5}.$$

   For a given $\epsilon > 0$, therefore, we can set $\epsilon' = \epsilon/5$, and choose N that works for the sequence $\{x_n\}$ and ϵ'. The same N does what's needed for $\{y_n\}$.

   (b) If $\{x_n\}$ is Cauchy, then $\{x_n\}$ converges, by Theorem 2.19. Now $\{5x_n + 2\}$ converges, too, by Theorem 2.5, page 83, and so $\{5x_n + 2\}$ must also be Cauchy, again by Theorem 2.19.

7. Let $\epsilon > 0$ be given and set $\epsilon' = \epsilon/42$. Because $\{a_n\}$ is Cauchy sequence there is some N such that $|a_n - a_m| < \epsilon'$ whenever $n > N$. This N works for the sequence $\{x_n\}$; details left to you.

8. (a) The sequence is monotone only if the coin falls heads (or tails) forever.

   (b) If the first six coin tosses show $HTHTHT$, then $a_6 = 21/32$, and it follows that all further a_n, and therefore the limit, must lie in the interval $[20/32, 22/32]$.

   (c) For every n, $|a_n| \leq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} < 2$.

9. Two numbers with the same first n decimal places can differ by no more than 10^{-n}.

10. (a) By Theorem 2.19, the result is equivalent to the (known) fact that the product of convergent sequences is convergent.

    (b) Suppose $|x_n| \leq M_1$ for all n and $|y_n| \leq M_2$ for all n. Now choose N_1 so

$$|x_n - x_m| < \frac{\epsilon}{2M_2} \quad \text{when} \quad n > m > N_1,$$

    and choose N_2 such that

$$|y_n - y_m| < \frac{\epsilon}{2M_1} \quad \text{when} \quad n > m > N_2.$$

    Now $N = \max\{N_1, N_2\}$ does what's needed, since if $n > m > N$, we have

$$\begin{aligned}|x_n y_n - x_m y_m| &\leq |x_n y_n - x_n y_m| + |x_n y_m - x_m y_m| \leq |x_n|\,|y_n - y_m| + |y_m|\,|x_n - x_m| \\ &\leq M_1\,|y_n - y_m| + M_2\,|x_n - x_m| < M_1 \frac{\epsilon}{2M_1} + M_2 \frac{\epsilon}{2M_2} = \epsilon.\end{aligned}$$

11. The key point is that, regardless of n,

$$\frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \cdots + \frac{1}{2^n} < \frac{1}{2^m}.$$

    This fact can be parlayed into a proof.

2.5 Series 101: Basic Ideas

1. (a) $S_n = 0$ for all n, and so the series converges to 0.
   (b) $S_n = 42n$ for all n; this diverges (to ∞).
   (c) The partial sum sequence $\{S_n\}$ has the form $-1,\ 0,\ -1,\ 0,\ \ldots$; this diverges.
   (d) $S_n = \frac{n(n+1)}{2}$ for all n; this diverges (to ∞).
   (e) The series is geometric, with $r = 0.99$, so $S_n = \frac{1-r^n}{1-r} = 100(1 - 0.99^n)$, and we see that $S_n \to 100$.
   (f) It is easy to show by induction that $S_n = \frac{n}{n+1}$. Thus $S_n \to 1$.

2. (a) Since $H_{2^n} > \frac{n}{2}$ for all n and the right-hand sequence diverges to infinity, we must have $H_{2^n} \to \infty$, too. Since $\{H_n\}$ has a divergent subsequence, $\{H_n\}$ must diverge, too.
   (b) Suppose toward contradiction that $H_n \to H$ for some finite number H. Since $\{H_{2n}\}$ is a subsequence, we'd have $H_{2n} \to H$, too. But we also have $H_{2n} > \frac{1}{2} + H_n$; taking limits of both sides gives $H \geq \frac{1}{2} + H$, which is absurd.
   (c) See almost any calculus text for the standard picture.

3. (a) $\sum_{k=1}^{\infty} \frac{1}{k+2^k}$ converges by comparison to the geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k}$.
   (b) $\sum_{k=1}^{\infty} \frac{k}{2k^2-1}$ diverges by comparison to the divergent series $\sum_{k=1}^{\infty} \frac{1}{2k}$. (Convince yourself that the comparison works.)
   (c) $\sum_{k=1}^{\infty} \frac{k}{3^k}$ converges by comparison to the convergent geometric series $\sum_{k=1}^{\infty} \frac{2^k}{3^k}$.
   (d) $\sum_{k=1}^{\infty} \frac{k^2+2}{3k^2+4}$ diverges by the nth term test: $\lim_{k\to\infty} \frac{k^2+2}{3k^2+4} = \frac{1}{3} \neq 0$.

4. (a) $S_n = 1 - \frac{1}{n+1}$, which clearly tends to the limit 1.
   (b) Here we get $S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$, so $S_n \to 1 + \frac{1}{2}$ as $n \to \infty$.
   (c) The pattern suggests that $S_n = 1 + \frac{1}{2} + \ldots \frac{1}{10} - -\frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{n+10}$. As $n \to \infty$, the last ten terms all tend to zero, so the limit is $S = 1 + \frac{1}{2} + \ldots \frac{1}{10} \approx 2.929$.

5. Let $\sum a_k$ and $\sum b_k$ be series, and let $\sum c_k$ be the "sum series," defined by $c_k = a_k + b_k$ for all k. Let A_n, B_n, and C_n denote the partial sums for these series. The key point is that—thanks to the commutative law for addition of *finitely* many numbers—$C_n = A_n + B_n$ for all n. It follows that if $A_n \to A$ and $B_n \to B$, then we must also have $C_n \to A + B$, which is what we wanted to prove.

6. (a) If $r = 1$, the series is just $a + a + a + \ldots$, and so $S_n = na$, which clearly diverges.
   If $r = -1$, the series is just $a - a + a - \ldots$, and so the successive S_n have the form $a,\ 0,\ a,\ 0,\ \ldots$; this, too, diverges.
   (b) i. The sequence $\{r^{n+1}\}$ is a subsequence of $\{r^n\}$, so both converge to the same place, L.
   ii. Let $x_n = r^n$. We know that $x_n \to L$; one of our theorems says $rx_n \to rL$.
   iii. The preceding parts say that $L = rL$; if $r \neq 0$, we must have $L = 0$.
   iv. If $0 \leq r < 1$, then $r^n \to 0$. (Hint: use a theorem about monotone sequences.) If $0 \leq r < 1$, then $r^{n+1} \leq r^n$ for all n, so the sequence is monotone decreasing and bounded below. Thus it converges, and the limit is zero by the preceding part.
   v. If $-1 < r \leq 0$, then $|r| < 1$, and so $|r|^n \to 0$ by the preceding part. Now we have $-|r|^n \leq r^n \leq |r|^n$, and so the middle limit is squeezed to zero.
   vi. If $|r| \geq 1$, then clearly $|r^n| \geq 1$ for all n. In particular, zero certainly isn't the limit. Thus the sequence diverges.

7. (a) It is enough to show (i) if $n > 1000$ then $S_n \geq S_{1000}$; (ii) if $n > 1000$ then $S_n \leq S_{1001}$. Claim (i) amounts to observing that *every* string of the form
$$\frac{1}{1001} - \frac{1}{1002} + \frac{1}{1003} - \cdots \pm \frac{1}{1000+n}$$
adds up to a *positive* result; group summands in *pairs* to see why. Claim (ii) holds because every string of the form
$$-\frac{1}{1002} + \frac{1}{1003} - \frac{1}{1004} + \cdots \pm \frac{1}{1000+n}$$
has *negative* sum; again, group summands in pairs to see why.

(b) For given $\epsilon > 0$ we can take any integer $N \geq 1/\epsilon$. Then $|S_N - S_{N+1}| = \frac{1}{N+1} < \epsilon$.

(c) For given $\epsilon > 0$ choose N as in the preceding part. Then for $n > m > N$, we have both S_n and S_m between S_N and S_{N+1}, and thus within ϵ of each other.

8. (a) Suppose $\sum d_k/10$ converges. By Theorem 2.22, $10 \sum d_k/10 = \sum d_k$ converges, too, which contradicts our hypotheses.

   (b) The product series $\sum c_k d_k$ may converge or diverge. If, say, $d_k = 1$ for all k and $c_k = 1/k^2$, then the product series converges. But if $d_k = k$ for all k and $c_k = 1/2^k$, then the product series diverges.

   (c) If A is constant then $\sum Ac_k$ converges by Theorem 2.22.

   (d) It is possible that $d_k \leq c_k$ for all k. This happens if, say, $d_k = -1$ and $c_k = 0$ for all k.

9. (a) Let A_n and B_n be the partial sums for $\sum a_k$ and $\sum b_k$. The hypothesis boils down to the fact that for $n > 42$ we have $B_n = A_n - A_{42} + B_{42}$. Thus, the sequences $\{A_n\}$ and $\{B_n\}$ differ (for large n) by an additive constant, and so both converge or both diverge.

   (b) We know (see the preceding part) that $B_n = A_n - A_{42} + B_{42}$. Because $b_k - a_k = k^2$, we must have $B_{42} - A_{42} = \sum_{k=1}^{4} 2k^2 = 25585$. Thus, $B_n = A_n + 25585$, and since $A_n \to 100$ we must have $B_n \to 25685$.

10. Since $\sum a_k$ converges we must have $a_k \to 0$, so there must exist K such that $0 < a_k < 1$ for $k > K$. Hence $a_k^2 < a_k$ for $k > K$. It follows that the series $\sum a_k^2$ converges by comparison to the series $\sum a_k$. (Note that the inequality $a_k^2 < a_k$ might not hold for $k \leq K$. Do you see why this doesn't matter?)

2.6 Series 102: Testing for Convergence and Estimating Limits

1. (a) $\sum_{k=1}^{\infty} \frac{1}{k+2^k}$ converges by comparison to $\sum_{k=1}^{\infty} \frac{1}{2^k}$.

   (b) $\sum_{k=1}^{\infty} \frac{k}{2k^2-1}$ diverges; one approach is limit comparison to $\sum_{k=1}^{\infty} \frac{1}{k}$.

   (c) $\sum_{k=1}^{\infty} \frac{k}{3^k}$ converges by comparison or limit comparison with the geometric series $\sum_{k=1}^{\infty} \frac{2^k}{3^k}$.

   (d) $\sum_{k=1}^{\infty} \frac{k^2+2}{3k^2+4}$ diverges; use the nth term test.

2. (a) $\sum_{k=1}^{\infty} \frac{100^k}{k!}$ converges by the ratio test.

   (b) $\sum_{k=1}^{\infty} \frac{\sin k}{1.0001^k}$ converges absolutely. The absolute value series converges by comparison to the geometric series $\sum_{k=1}^{\infty} \frac{1}{1.0001^k}$.

   (c) $\sum_{k=1}^{\infty} \frac{k^2+k+1}{k^3+k^2+k+1}$ diverges; one approach is limit comparison to $\sum_{k=1}^{\infty} \frac{1}{k}$.

   (d) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln\ln(k+2)}$ converges by the alternating series test.

3. (a) The ratio $\frac{a_k^2}{a_k} = a_k$ tends to zero (by the kth term test), and so $\sum a_k^2$ converges.

   (b) If $\sum a_k = \sum \frac{1}{\sqrt{k}}$, then $\sum a_k^2$ diverges. If $\sum a_k = \sum \frac{1}{k}$, then $\sum a_k^2$ converges.

4. Suppose toward contradiction that $\sum a_k$ converges. Since $a_k/b_k \to \infty$ (see the hint), we have $b_k/a_k \to 0$, and the limit comparison test implies that $\sum b_k$ converges, a contradiction.

5. If $\sum b_k$ converges to B, then $\sum a_k$ converges to $B - \sum_{k=1}^{K} b_k + \sum_{k=1}^{K} a_k$. This is similar to a problem in the preceding section, where $K = 42$.

6. One can show easily (e.g., by induction) that $k! > 2^k$ for $k > 3$. Thus, $1/k! < 1/2^k$ for all $k > 3$, and hence $\sum 1/k!$ converges by comparison to $\sum 1/2^k$.

7. Note that the sum
$$S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}.$$
Since $\sqrt{n} \to \infty$, we must have $S_n \to \infty$, too.

9. (a) The series converges by comparison to $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$, which converges to 2.
   (b) We have $R_{10} = \sum_{k=11}^{\infty} \frac{1}{2^{k-1}+1} < \sum_{k=11}^{\infty} \frac{1}{2^{k-1}} = \frac{1}{2^9}$.
   (c) We have $S = S_{10}+R_{10} < 1.26255+\frac{1}{2^9} \approx 1.26450$. This means that S lies somewhere in the interval $[1.26255, 1, 26450]$.

10. (a) The series converges absolutely by comparison to $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$, which converges to 2.
   (b) Let S be the limit. The alternating series theorem says that $|S_n - S| < \frac{1}{2^n+1}$. The right side is the size of the $(n+1)$th term, which is less than 0.001 if $n \geq 10$. This means that the limit lies between $S_{10} \approx 0.29365$ and $S_{11} \approx 0.29462$.
   (c) The partial sum S_{100} can differ from S by no more than the size of the next term, which is $\frac{1}{2^{100}+1}$. This corresponds to something like 30-digit accuracy.

11. (a) See your favorite calculus text.
   (b) We need to choose n so $R_n < 0.001$. By the given inequality, this holds if n is large enough so that $\int_n^{\infty} \frac{dx}{x^3} < 0.001$. A calculus calculation shows that $\int_n^{\infty} \frac{dx}{x^3} = \frac{1}{2n^2}$, and the last quantity is less than 0.001 if $n \geq 23$. This means that $S_{23} \approx 1.2012$ is within 0.001 of the true answer.

3.1 Limits of Functions

1. Both parts are easy exercises with the definition. In (a) we can choose *any* δ for given ϵ. In (b) we can choose $\delta = \epsilon$.

2. (a) $\lim_{x\to 1}(2x+3) = 5$. Note that $|f(x) - 5| = 2, |x - 1|$. It follows that for given $\epsilon > 0$ we can set $\delta = \epsilon/2$.
   (b) Factoring the numerator gives $\lim_{x\to -1} \dfrac{x^2-1}{x+1} = \lim_{x\to -1}(x-1) = -2$. (For given $\epsilon > 0$ we can set $\delta = \epsilon$.)
   (c) Factoring the numerator gives $\lim_{x\to 1} \dfrac{x^2+x-2}{x-1} = \lim_{x\to 1}(x+2) = 3$. (For given $\epsilon > 0$ we can set $\delta = \epsilon$.)
   (d) The limit is zero. To see why, note that

$$|f(x) - 0| = \left| x\,\frac{2+\sin x}{3-\cos x} \right| \leq |x|\,\frac{3}{2}.$$

   (The factor $3/2$ comes from considering the possible sizes of numerator and denominator.) It follows that for given $\epsilon > 0$, we can set $\delta = 2\epsilon/3$.

3. (a) For $f(x) = x^2$, we have $\lim_{x\to 42} f(x) = \lim_{x\to 42} x^2 = 42^2 = f(42)$, so the condition does hold at $a = 42$.
   (b) For $f(x) = x^2$ and *any* input a, we have $\lim_{x\to a} f(x) = \lim_{x\to a} x^2 = a^2 = f(a)$, so the condition holds at every a.
   (c) Since the function $f(x) = \frac{x^2-4}{x-2}$ is not defined at $a = 2$, the condition can't possibly hold.
   (d) We have $\lim_{x\to 3} f(x) = \lim_{x\to 3} \dfrac{x^2-4}{x-2} = \dfrac{3^2-4}{3-1} = f(3)$, as desired.
   (e) Note that $a = 0$ is an endpoint of the domain. Also, we know $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \sqrt{x} = 0 = f(0)$, so the condition holds in the one-sided sense.
   (f) We have $\lim_{x\to a} f(x) = \lim_{x\to a} |x| = |a| = f(a)$ for all a.
   (g) Theorem 3.3, Page 138, assures that $\lim_{x\to a} f(x) = \lim_{x\to a}(1 + \pi x + ex^2 + \pi e x^3) = 1 + \pi a + ea^2 + \pi e a^3 = f(a)$, so the condition holds for all a.

4. Let $\{x_n\}$ be any sequence with $x_n \to a$ and $x_n \neq a$ for all n. Then $\{f(x_n)\}$, $\{g(x_n)\}$, and $\{h(x_n)\}$ are all *sequences*, and the hypotheses imply that (i) $f(x_n) \leq g(x_n) \leq h(x_n)$ for all n; and (ii) $f(x_n) \to L$ and $h(x_n) \to L$. Now the sequence version of the squeezing theorem implies that $g(x_n) \to L$, too. Since this applies to *any* sequence $\{x_n\}$ of the given type, Lemma 3.2 implies that $\lim_{x\to a} g(x) = L$, as desired.