CHAPTER 2

SECTION 2.1

1. On our TI–81, $y = .3333333333$. If $n = 37$, then the TI-81 yields $|B^{[n]}(x) - B^{[n]}(y)| = |2/3 - .085368176| > 1/2$. Moreover, 37 is the smallest value of n such that $|B^{[n]}(x) - B^{[n]}(y)| > 1/2$.

2. Let x be an arbitrary number in $[0, 1]$. Arbitrarily close to x are a dyadic rational $y \neq 0$ and an irrational number z. Since y is dyadic, it follows from Exercise 1.3.14 that the iterates of y are eventually 1. Thus there is a positive integer m such that if $n \geq m$, then $B^{[n]}(y) = 1$. Since z is irrational, and hence not dyadic, Exercise 1.3.18 tells us that for some $n_0 \geq m$, $B^{[n_0]}(x) < 1/2$. Consequently $|B^{[n_0]}(y) - B^{[n_0]}(z)| > 1/2$. It follows that either $|B^{[n_0]}(x) - B^{[n_0]}(y)| > 1/4$ or $|B^{[n_0]}(x) - B^{[n_0]}(z)| > 1/4$. As a result, B has sensitive dependence at the arbitrary number x, so B has sensitive dependence.

3. a. If x is not a dyadic rational, then the iterates of x are not dyadic rationals either, so that $(B^{[n]})'(x)$ exists and $(B^{[n]})'(x) = 2^n$. It follows that

$$\lim_{n\to\infty} \frac{1}{n} \ln \left|(B^{[n]})'(x)\right| = \lim_{n\to\infty} \frac{1}{n} \ln 2^n = \ln 2$$

Therefore $\lambda(x) = \ln 2$ whenever x is not a dyadic rational.

b. Let $f(x) = B^{[2]}(x)$, so that $f^{[n]} = B^{[2n]}$. If x is not a dyadic rational, then $(f^{[n]})'(x) = 2^{2n}$ for all n. Therefore

$$\lim_{n\to\infty} \frac{1}{n} \ln 2^{2n} = 2 \ln 2$$

4. In order that $B_\mu(x)$ be in $[0, 1]$ for all x in $[0, 1]$, it is necessary that $0 \leq \mu \leq 1$.

a. Suppose that $0 \leq \mu < 1/2$. If $0 \leq x \leq 1/2$, then $B_\mu(x) = 2\mu x < 1/2$; similarly, if $1/2 < x \leq 1$, then $B_\mu(x) = \mu(2x - 1) < (1/2)(2x - 1) = x - 1/2 \leq 1/2$. Thus if x is any number in $[0, 1]$, then $B_\mu(x)$ lies in $[0, 1/2)$, so that for any positive integer $n \geq 2$,

$$\left|B_\mu^{[n]}(x)\right| = \left|B_\mu^{[n-1]}(B_\mu(x))\right| = (2\mu)^{n-1}\,|B_\mu(x)|$$

Since $0 \leq 2\mu < 1$, it follows that the iterates of x converge to 0. Therefore B_μ does not have sensitive dependence. Next, if $\mu = 1/2$, then for any x and y in $[0, 1/2]$ we have $|B_\mu(x) - B_\mu(y)| = |x - y|$, so B_μ does not have sensitive dependence. Finally, assume that $1/2 < \mu < 1$. If x and y are both in $[0, 1/2]$ or both in $[1/2, 1]$, then $|B_\mu(x) - B_\mu(y)| = 2\mu|x - y|$, so that while they remain in the same half of the interval $[0, 1]$, the iterates of x and y separate from one another by a factor of 2μ, which is greater than 1. It follows that if x and y are distinct, then for some n, $B_\mu^{[n]}(x)$ will lie in the interval $(3/8, 1/2)$ and $B_\mu^{[n]}(y)$ will lie in $(1/2, 5/8)$ (or vice versa). Without loss of generality, suppose that $3/8 < B_\mu^{[n]}(x) < 1/2 < B_\mu^{[n]}(y) < 5/8$. Then $3/4 < B_\mu^{[n+1]}(x) < 1$ and $0 < B_\mu^{[n+1]}(y) < 1/4$, so that $|B_\mu^{n+1}(x) - B_\mu^{n+1}(y)| \geq 3/4 - 1/4 = 1/2$. Thus B_μ has sensitive dependence if $1/2 < \mu < 1$.

b. At points where $\lambda(x)$ exists we find that

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln B_\mu'(x_k) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln (2\mu) = \ln (2\mu)$$

5. Let $\varepsilon > 0$. Since f is continuous, there is a $\delta > 0$ such that if $|x - p| < \delta$, then $|f^{[k]}(x) - f^{[k]}(p)| < \varepsilon$ for $k = 1, 2, \ldots, n$. Since by hypothesis, $|(f^{[n]})'(p)| < 1$, we know that p is an attracting fixed point of $f^{[n]}$. As a result, there is a δ^* such that $0 < \delta^* \le \delta$ and such that if $|x - p| < \delta^*$, then $|f^{[n]}(x) - f^{[n]}(p)| < |x - p|$. Consequently for such an x, $|f^{[k]}(x) - f^{[k]}(p)| < \varepsilon$. This means that f cannot have sensitive dependence on initial conditions at p.

6. a. Let $x_0 = x$, and let $x_n = f^{[n]}(x_0)$, for $n = 1, 2, \ldots$. If $x_N = p$, then

$$\begin{aligned}\lambda(x) &= \lim_{n\to\infty} \frac{1}{n} \ln \left|(f^{[n]})'(x_0)\right| = \lim_{n\to\infty} \frac{1}{n} \ln |f'(x_0)f'(x_1)\cdots f'(x_{N-1})f'(x_N)\cdots f'(x_{n-1})| \\ &= \lim_{n\to\infty} \frac{1}{n} \ln |f'(x_0)f'(x_1)\cdots f'(x_{N-1})| + \lim_{n\to\infty} \frac{n-N}{n} \ln |f'(p)| \\ &= 0 + \ln |f'(p)| = \ln |f'(p)|\end{aligned}$$

b. Let $x_0 = x$, and let $x_n = f^{[n]}(x_0)$, for $n = 1, 2, \ldots$. If $x_N = p$, then $x_{N+1} = q$, $x_{N+2} = r$, $x_{N+3} = p$, etc. It follows that

$$\begin{aligned}\lambda(x) &= \lim_{n\to\infty} \frac{1}{n} \ln \left|(f^{[n]})'(x)\right| = \lim_{n\to\infty} \frac{1}{n} \ln |f'(x_0)f'(x_1)\cdots f'(x_{N-1})f'(x_N)\cdots f'(x_{n-1})| \\ &= \lim_{n\to\infty} \frac{1}{n} \ln |f'(x_0)f'(x_1)\cdots f'(x_{N-1})| + \lim_{n\to\infty} \frac{n-N}{3n} \ln |f'(p)f'(q)f'(r)| \\ &= \frac{1}{3} \ln |f'(p)f'(q)f'(r)|\end{aligned}$$

7. a. From Theorem 1.16 we know that the 2-cycle $\{q_\mu, r_\mu\}$ attracts every x in $(0, 1)$ except the fixed point $p_\mu = 1 - 1/\mu$ and its preimages. Suppose first that x is attracted to the 2-cycle. Let $\varepsilon > 0$. Since the iterates of x converge to the 2-cycle, there exists an N such that

$$\left|\ln |Q'_\mu(q_\mu)Q'_\mu(r_\mu)| - \ln |Q'_\mu(x_{2k})Q'_\mu(x_{2k+1})|\right| < \varepsilon \text{ for all } k \ge N$$

Since it is only the "tail end" of the iterates of x that affects $\lambda(x)$, we know that $\lambda(x) = \lambda(x_N)$. If $n > N$, then by the preceding inequality, we find that

$$\begin{aligned}\lambda(x_N) &= \lim_{n\to\infty} \frac{1}{2n} \sum_{k=0}^{2n-1} \ln |Q'_\mu(x_k)| = \lim_{n\to\infty} \frac{1}{2n} \sum_{j=0}^{n-1} \ln |Q'_\mu(x_{2j})Q'_\mu(x_{2j+1})| = \lim_{n\to\infty} \frac{1}{2} \ln |Q'_\mu(q_\mu)Q'_\mu(r_\mu)| \\ &= \frac{1}{2} \ln |\mu^2 - 2\mu - 4| < 0\end{aligned}$$

b. If x is eventually the fixed point $p_\mu = 1 - 1/\mu$, then by Exercise 6(a), $\lambda(x) = \ln |Q'_\mu(p)| = \ln |\mu - 2|$.

8. a. $Q_4^{[n]}(1/2) = 0$ for $n \ge 2$.

b. We find that $Q_4^{[7]}(0.51) \approx 0.91785 > 1/2$, $Q_4^{[10]}(0.501) \approx 0.72891 > 1/2$, and $Q_4^{[13]}(0.5001) \approx 0.70807 > 1/2$. If $y = 0.51, 0.501$, or 0.5001, then by part (a), $|Q_4^{[n]}(1/2) - Q_4^{[n]}(y)| = Q_4^{[n]} > 1/2$ for an appropriate n.

SECTION 2.2

1. Let $B_n = [n, \infty)$ for each positive integer n. Then each B_n is closed, and the sequence is nested. However, the intersection of the B_n is void.

2. Let $B_n = (0, 1/n)$ for each positive integer n. Then each B_n is bounded, and the sequence is nested. However, the intersection of the B_n is void.

3. Let (a, b) be an arbitrary open interval in $[0, 1]$. Then there exists a positive integer n such that $1/2^n < b - a$. Now (a, b) contains at least one (dyadic) rational $k/2^n$, because for each k, $(k+1)/2^n - k/2^n = 1/2^n < b - a$. Thus the dyadic rationals, and hence the rationals, are dense in $[0, 1]$.

4. a. Since $h(x) = \overline{1}\cdots$, it follows that $h(T(x)) = \overline{1}\cdots$, so that $x = T(x)$. Thus x is a fixed point. But the only fixed points of T are 0 and 2/3. Since $h(0) = \overline{0}\cdots$ and $h(2/3) = \overline{1}\cdots$, the desired number is 2/3.

   b. Since $h(x) = \overline{10}\cdots$, it follows that x is a period-2 point. The two period-2 points of T are 2/5 and 4/5. Since $h(2/5) = \overline{01}\cdots$ and $h(4/5) = \overline{10}\cdots$, we know that the desired number is 4/5.

5. a. Suppose that $h(x) = 11\overline{0}\cdots$. Then $h(T(x)) = 1\overline{0}\cdots$. Since $h(1) = 1\overline{0}\cdots$ and h is one-to-one, we conclude that $T(x) = 1$. This means that $x = 1/2$. However, $h(1/2) = 01\overline{0}\cdots$. This contradiction proves that there is no x such that $h(x) = 11\overline{0}\cdots$.

   b. If there were an x such that $h(x) = x_0x_1x_2\cdots x_n$, then it would follow that $h(T^{[n+1]}(x)) = 11\overline{0}\cdots$. By part (a), this is impossible.

6. By Theorem 2.9 we know that there is an element s in D. If U is an arbitrary nonempty interval in $[0, 1]$, then there is some iterate, say y, of s that is in U. Since s is irrational, so is y. Moreover, y is in D because the collection of iterates of y differs from that of s by merely a finite set of points. Thus D is dense in $[0, 1]$.

7. a. Let $s = 0\ 1\ 001\ 0001\ 00001\ 000001\cdots$, so that each block of zeros has one more zero than the previous block. Since s is not a finite sequence, Theorem 2.8 assures us that there is an x in $[0, 1]$ such that $h(x) = s$. Since s never repeats, x is irrational. Notice that no iterate of x begins $11\cdots$. This means that no iterate x lies in $(1/2, 3/4)$. In other words, x is an irrational number without a dense orbit.

   b. Let D_0 consist of all x in $[0, 1]$ such that for some n, $h(T^{[n]}(x)) = s$, where s is the sequence defined in part (a). Then D_0 is dense in $[0, 1]$, and $D_0 \subseteq D^*$. Consequently D^* is dense in $[0, 1]$.

8. a. Assume that $h(x) = x_0x_1x_2\cdots$, and that

$$z = \frac{g(x_0)}{2} + \frac{g(x_0, x_1)}{2^2} + \frac{g(x_0, x_1, x_2)}{2^3} + \cdots$$

We will show that $h(z) = x_0x_1x_2\cdots = h(x)$. Since h is one-to-one, it will follow that $x = z$.

Let $0 \le z \le 1/2$, and suppose that $x_0 = 1$. By the definition of g, this would imply that $g(x_0\, x_1, \ldots, x_k) = 0$ for all $k \ge 1$. This would mean that $x_0x_1x_2\cdots = 11\overline{0}\cdots$, which is impossible by Exercise 5. Therefore $x_0 = 0$, so that the first term of $h(z)$ is x_0. Now suppose that $1/2 < z \le 1$. By the definitions of g and z, $x_0 \pmod 2 = g(x_0) = 1$, so that $x_0 = 1$. Therefore the first term of $h(z)$ is $x_0 = 1$. We conclude that for any z in $[0, 1]$, the first term of $h(z)$ is the same as the first term of $h(x)$.

Next we will show that

$$T(z) = \frac{g(x_1)}{2} + \frac{g(x_1, x_2)}{2^2} + \frac{g(x_1, x_2, x_3)}{2^3} + \cdots$$

On the one hand, suppose that $0 \le z \le 1/2$. Then $x_0 = 0$ and $T(z) = 2z$. Therefore

$$T(z) = g(0) + \frac{g(0, x_1)}{2} + \frac{g(0, x_1, x_2)}{2^2} + \cdots = \frac{g(x_1)}{2} + \frac{g(x_1, x_2)}{2^2} + \cdots$$

On the other hand, suppose that $1/2 < z \le 1$. Then $x_0 = 1$ and $T(z) = 2 - 2z$, so that

$$T(z) = 2 - g(1) - \frac{g(1, x_1)}{2} - \frac{g(1, x_1, x_2)}{2^2} - \cdots$$

Since $g(1, x_1, x_2, \ldots, x_k) = 1 - g(x_1, \ldots, x_k)$, we can write $T(z)$ as follows:

$$T(z) = 2 - 1 - \frac{1 - g(x_1)}{2} - \frac{1 - g(x_1, x_2)}{2^2} - \frac{1 - g(x_1, x_2, x_3)}{2^3} - \cdots$$

$$= \left(2 - 1 - \frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} - \cdots\right) + \left(\frac{g(x_1)}{2} + \frac{g(x_1, x_2)}{2^2} + \frac{g(x_1, x_2, x_3)}{2^3} + \cdots\right)$$

$$= \frac{g(x_1)}{2} + \frac{g(x_1, x_2)}{2^2} + \frac{g(x_1, x_2, x_3)}{2^3} + \cdots$$

Repeating our earlier argument involving x_0, we find that if $0 \le T(z) \le 1/2$, then $x_1 = 0$, whereas if $1/2 < T(z) \le 1$, then $x_1 = 1$. Thus the second term of $h(z)$ is the same as the second term of $h(x)$. Continuing with $T^{[2]}(z)$, $T^{[3]}(z)$, …, we find that one by one the terms of $h(z)$ are the same as those of $h(x)$. In other words, $h(z) = h(x)$. Since h is one-to-one, we conclude that $z = x$, which completes the solution of part (a).

b. By part (a), it suffices to compute the partial sum

$$\frac{g(0)}{2} + \frac{g(0, 1)}{4} + \frac{g(0, 1, 0)}{8} + \frac{g(0, 1, 0, 0)}{16} + \frac{g(0, 1, 0, 0, 0)}{32} + \frac{g(0, 1, 0, 0, 0, 1)}{64} + \frac{g(0, 1, 0, 0, 0, 1, 1)}{128}$$

of z corresponding to x_t, because the remaining terms add up to no more than $1/100$ because $0 \le g \le 1$. Now the partial sum of the series is $1/4 + 1/8 + 1/16 + 1/32 + 1/128 = 61/128$. Therefore

$$\frac{61}{128} < x_t < \frac{62}{128}, \text{ or equivalently, } 0.476563 < x_t < 0.484375$$

9. We obtain $Q_4^{[940]}(0.28) \approx 0.62985$.

10. Our computer shows that $|Q_4^{[71]}(.45) - 1| < .001$.

11. By Exercise 2.1.2, B has sensitive dependence. By Exercise 1.3.16, every rational k/p in $[0, 1)$ such that p is a positive odd integer is periodic. Thus the periodic points of B are dense in $[0, 1]$. The same kind of association k can be made between x in $[0, 1]$ and a sequence of 0's and 1's as is made for T. The number x such that $k(x) = 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001 \cdots$ has a dense orbit.

12. a. Let

$$h_0(x) = x_0x_1x_2\cdots, \text{ where } \begin{cases} 0 \text{ if } & 0 \le f^{[n]}(x) \le 1/3 \\ 1 \text{ if } & 1/3 < f^{[n]}(x) \le 2/3 \\ 2 \text{ if } & 2/3 < f^{[n]}(x) \le 1 \end{cases}$$

The proof that $h_0 : [0, 1] \to A_0$ is one-to-one and onto is similar to the corresponding proof for h.

b. Our solution will be patterned after the solutions of Exercises 1.3.15 and 1.3.16. Presently we will prove that if $0 < x < 1$ and x is rational, then x is eventually periodic. Since 0 and 1 are eventually periodic, let us assume that $x = k/p$ is in reduced form, with $0 < k < p$. Then $f(x) = 3x - j$, for an appropriate nonnegative integer j. Therefore $f(x) = m/r$, where r divides p and $0 < m \le r$. However, there are only finitely many distinct such possibilities. Consequently eventually the iterates of x must repeat. This means that x is eventually periodic.

Now assume that $x = k/p$, where p is prime to 3. To show that x is periodic, let S_p denote the collection of all k/p in reduced form, where $0 < k < p$. Then the iterates of x are in S_p because p is prime to 3. Now if $f(x) = f(y)$ and $x < y$, then $3x = 3y - 1$ or $3x = 3y - 2$, in which case, $y = (3k + 1)/(3p)$ or $y = (3k + 2)/(3p)$ in reduced form. Either way, y is not in S_p. Thus x in S_p can be the iterate of only one number in S_p. Since x in S_p is eventually periodic by the preceding paragraph, we conclude that the orbit of x is S_p, so that x is periodic.

c. By a proof similar to that given in Exercise 2.1.2, it follows that f has sensitive dependence on initial conditions. For the function h_0 defined in part (a), let x be the number in $[0, 1]$ such that

$$h_0(x) = 0\;1\;2\;00\;01\;02\;10\;11\;12\;20\;21\;22\;000\;001\;002\cdots$$

Since the orbit of $h_0(x)$ is dense in A_0, one can show that the orbit of x is dense in $[0, 1]$.

13. a. Let n be an odd integer. By Theorem 1.15, k/n is periodic for all even integers $0 < k < n$. The orbit of k/n is contained in the set $S_n = \{2/n, 4/n, \ldots, (n-1)/n\}$. Thus S_n is a disjoint union of cycles. The cycles may have different lengths, as in the case for $n = 9$. But if we now assume additionally that n is prime, then we can show that the period of each element must divide the cardinality of S_n, which is $(n-1)/2$.

To prove this, we will show that each element of S_n is a fixed point of $T^{[(n-1)/2]}$. One can describe the graph of $T^{[(n-1)/2]}$ explicitly, and prove that the fixed points of $T^{[(n-1)/2]}$ have the form $2j\,(2^{(n-1)/2} \pm 1)$ (see the solution of Exercise 1.4.2, for example.) Since n is prime, we know that an element of S_n is of this form if and only if n divides $2^{(n-1)/2} \pm 1$, or equivalently, $2^{(n-1)/2} \equiv \pm 1 \pmod n$. Now consider the set $P = \{1, 2, 3, \ldots, n-1\}$, and let $f : P \to P$ be the function defined by $f(x) = 2x \pmod n$. Now f is invertible on P, since if k is even, then $f(k/2) = k$, while if k is odd, then $f((n + k)/2) = k$. Therefore each element of P is periodic. If C is a cycle in P, then so is mC, which is the set obtained from C by multiplying all of its elements by $m \bmod n$. Since n is prime, it follows that C and mC have the same cardinality. Hence all the cycles in P have the same cardinality, and this cardinality divides $n - 1$. In particular, we have

$$2 = f^{[n-1]}(2) \equiv 2^n \pmod n$$

Therefore $2^{n-1} \equiv 1 \pmod n$, so that $2^{(n-1)/2} \equiv \pm 1 \pmod n$. This completes the proof that each element of S_n is a fixed point of $T^{[(n-1)/2]}$, and hence has a period that is a divisor of $(n-1)/2$. If we now also assume that $(n-1)/2$ is prime, then it follows that S_n consists of a single orbit.

b. If there are infinitely many prime pairs $\{n, (n-1)/2\}$, then by part (a) there exist orbits having arbitrarily many elements, all spread evenly throughout the interval $[0, 1]$. Thus if U and V are any two nonempty open intervals in $[0, 1]$, then there exists an orbit having at least one point in each interval. This implies that T is transitive.

SECTION 2.3

1. In Theorem 2.16 let $a = -\mu$, $r = -\mu$, $b = 2 - \mu$, $s = \mu$, and $t = 0$. Then $c = 0$, $d = 1$, and $e = (-1/\mu) + 1$. Therefore $h(x) = x - (1/\mu) + 1$. Notice that $h^{-1}(x) = x + (1/\mu) - 1$, so that $h^{-1}(0) = (1/\mu) - 1$ and $h^{-1}(1) = 1/\mu$. Since h is linear on L and since its range must be the domain of Q_μ which is $[0, 1]$, it follows that $L = [(1/\mu) - 1, 1/\mu]$.

2. In Theorem 2.16 let $a = -1$, $b = 0$, $r = -\mu$, $s = \mu$, and $t = 0$. Then $c = \mu^2/4 - \mu/2$, $d = 1/\mu$, and $e = 1/2$. Therefore $h(x) = x/\mu + 1/2$. Notice that $h^{-1}(x) = \mu x - \mu/2$, so that $h^{-1}(0) = -\mu/2$ and $h^{-1}(1) = \mu/2$. Since h is linear on L and its range must be the domain of Q_μ, which is $[0, 1]$, it follows that $L = [-\mu/2, \mu/2]$.

3. a. In Theorem 2.16, let $a = -\mu$, $b = \mu$, $c = 0$, $r = -C$, $s = 0$, and $t = 1$. Then $0 = (\mu^2 - 2\mu - 4C)/(-4\mu)$, so that $C = \mu^2/4 - \mu/2$. Thus

$$h(x) = \frac{4}{2-\mu}\,x + \frac{2}{2-\mu}$$

so that Q_μ and F_C are conjugates.

   b. Since 0 is a critical point of F_C, Singer's Theorem tells us that it suffices to determine iterates of 0.

   c. If $C = 7/4$, then since $C = \mu^2/4 - \mu/2$, we find that $7/4 = \mu^2/4 - \mu/2$, which is equivalent to $\mu^2 - 2\mu - 7 = 0$. The solutions of this equation are $\mu = 1 \pm 2\sqrt{2}$. Since $\mu > 0$, we conclude that $\mu = 1 + 2\sqrt{2}$. Since h found in part (a) is linear and is increasing, it follows from Theorem 2.13(ii) that if $\mu > 1 + 2\sqrt{2}$, then Q_μ has a 3-cycle.

4. From Theorem 2.16 it follows that if $c = C$, then K_c is conjugate to F_C via h, where $h(x) = x/C$.

5. a. A 4-cycle of T is $\{2/15, 4/15, 8/15, 14/15\}$. By Example 2 and Theorem 2.13(ii) it follows that a 4-cycle of Q_4 is

$$\{\sin^2(\frac{\pi}{2}\cdot\frac{2}{15}),\ \sin^2(\frac{\pi}{2}\cdot\frac{4}{15}),\ \sin^2(\frac{\pi}{2}\cdot\frac{8}{15}),\ \sin^2(\frac{\pi}{2}\cdot\frac{14}{15})\}$$

which is approximately $\{0.043227, 0.165435, 0.552264, 0.989074\}$.

   b. From Exercise 1.4.2, we know that $\{2/11, 4/11, 8/11, 6/11, 10/11\}$ is a 5-cycle of T. By Example 2 and Theorem 2.13(ii), and by the method used in part (a), we find that a 5-cycle of Q_4 is approximately

$$\{0.079373, 0.292292, 0.827430, 0.571157, 0.979746\}$$

6. From Section 1.5 we know that if $3 < \mu < 3.25$, then Q_μ has a 2-cycle, denoted by $\{q_\mu, r_\mu\}$. By Exercise 1 and Theorem 2.13(ii) it follows that $\{h^{-1}(q_\mu), h^{-1}(r_\mu)\}$ is a 2-cycle for f_μ, where $h^{-1}(x) = x + (1/\mu) - 1$. From the formulae for q_μ and r_μ on page 48 we find that

$$h^{-1}(q_\mu) = -\frac{1}{2} + \frac{3}{2\mu} - \frac{1}{2\mu}\sqrt{(\mu-3)(\mu+1)} \quad \text{and} \quad h^{-1}(r_\mu) = -\frac{1}{2} + \frac{3}{2\mu} + \frac{1}{2\mu}\sqrt{(\mu-3)(\mu+1)}$$

7. Suppose there exists a polynomial h, given by $h(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, such that $h \circ f = g_\mu \circ h$. This would mean that

$$a_n x^{3n} + \cdots + a_1 x^3 + a_0 = (a_n x^n + \cdots + a_0) - \mu(a_n x^n + \cdots + a_0)^3$$

Equating the highest-power terms yields $a_n x^{3n} = -\mu a_n^3 x^{3n}$ for all x. Since $a_n \neq 0$ by hypothesis, this means that $1 = -\mu a_n^2$, which implies that $\mu < 0$. However, this contradicts the hypothesis that $\mu > 0$. Consequently no such polynomial h exists.

8. If Q_4 and T were linearly conjugate via h, then h would be linear with domain and range both equal to $[0, 1]$. In that case, $h(x) = x$ or $h(x) = 1 - x$, neither of which conjugates Q_4 to T. (We note that any function linearly conjugate to a linear function must be linear. As a result, any function linearly conjugate to T must be piecewise linear, which Q_4 is not.)

9. Since T_λ is piecewise linear and Q_μ is not piecewise linear, the same reasoning given in the solution of Exercise 8 applies to show that Q_μ and T_λ cannot be linearly conjugate to one another.

10. Let $H_0 = H \circ h$. Then H_0 is linear since both H and h are linear. Moreover,

$$H_0 \circ f = H \circ h \circ f = H \circ g \circ h = k \circ H \circ h = k \circ H_0$$

Thus f and k are conjugate via H_0.

11. Since h is linear, so is h^{-1}. Moreover, since $h \circ f = g \circ h$, we find that $f = h^{-1} \circ h \circ f = h^{-1} \circ g \circ h$, so that $f \circ h^{-1} = h^{-1} \circ g \circ h \circ h^{-1} = h^{-1} \circ g$.

12. a. By the Intermediate Value Theorem (or by Lemma 1 on p. 63), $f(U)$ is an interval. Suppose that y is a boundary point of $f(U)$, say the largest number in $f(U)$. Since f is a homeomorphism, there is an x in U such that $f(x) = y$. Since U is an open interval, there is an interval of the form $(x - \varepsilon, x + \varepsilon)$ contained in U. If $f(x - \varepsilon/2) \leq y$ and $f(x + \varepsilon/2) \leq y$, then by the Intermediate Value Theorem, f would not be one-to-one and hence would not be a homeomorphism. Consequently either $f(x - \varepsilon/2) > y$ or $f(x + \varepsilon/2) > y$. Either way y is not the largest number in $f(U)$, which means that $f(U)$ has no largest number. Similarly, $f(U)$ has no smallest number. Thus $f(U)$ is an open interval.

    b. Suppose that A is dense in J, and let V be an open interval in K. Since f^{-1} is a homeomorphism, part (a) tells us that $f^{-1}(V)$ is an open interval in J. The fact that A is dense in J implies that there is a number a in $A \cap f^{-1}(V)$. Therefore $f(a)$ is in $f(A) \cap V$. Since V is arbitrary, we conclude that $f(A)$ is dense in K.

13. For any μ, let $h(x) = x - x_\mu$, so $h^{-1}(x) = x + x_\mu$. If $g_\mu(x) = (h \circ f_\mu \circ h^{-1})(x)$, then 0 is a fixed point of g_μ.

14. a. Let $h(x) = 3x$. Then $(h \circ f)(x) = h(f(x)) = h(g(3x)/3) = g(3x) = (g \circ h)(x)$. Thus f is conjugate to g via h.

    b. Since $F(x) \geq 2/3$ for $0 \leq x \leq 1/3$, and $F(x) \leq 1/3$ for $2/3 \leq x \leq 1$, and since F is continuous and decreasing on $[1/3, 2/3]$, the unique fixed point of F lies in $(1/3, 2/3)$. Since the slope of F at the fixed point is -3, the fixed point is repelling and after a certain iterate, all higher iterates of any point near the fixed point are in $[0, 1/3] \cup [2/3, 1]$. If x is in $[0, 1/3]$, then $F(x)$ is in $[2/3, 1]$ and $F^{[2]}(x)$ is in $[0, 1/3]$. Moreover, $F^{[2]}(x) = F(2/3 + f(x)) = f(x)$, so that by induction, for x in $[0, 1/3]$ we have $F^{[2n]}(x) = f^{[n]}(x)$ for $n = 1, 2, \ldots$. Thus any periodic point of F is a periodic point of f of half the period. But since f is conjugate to g, any periodic point of F corresponds to a periodic point of g of half the period.

SECTION 2.4

1. a. Let S be a finite set, and $\{x_n\}_{n=1}^{\infty}$ a sequence in S converging to a point x. If x is not in S, then let c be the minimum distance between x and every element in S. Thus $c > 0$, and no points of S are within c of x. But this is impossible since x is the limit of the sequence. Thus x is in S, so that S is closed.

   b. Let S be a finite union of closed intervals. If two closed intervals intersect, then their union is a closed interval. Thus we may assume that S is a finite union of disjoint closed intervals, which are separated from one another at least by some positive distance c. If $\{x_n\}_{n=1}^{\infty}$ is a sequence in S that converges to x, then it follows that for some k, $\{x_n\}_{n=k}^{\infty}$ lies in a single closed interval I of S. Since I is closed, the limit of the second sequence, which is also x, lies in I, and hence in S. Thus S is closed.

2. a. Suppose that r is in $(0, 1)$, and that r has a decimal expansion that is repeating. Then $r = 0.g\bar{h}\cdots$, where g is some finite block of initial digits and h is a finite block of repeating digits. The simplest case occurs when $h = 0$, so that the decimal expansion terminates. Then $r = g/10^k$, where k is the length of the block g. Thus r is a rational number. Next, consider the case where g is empty, so that the decimal expansion begins repeating right away. In this case,

$$r = \frac{h}{10^j} + \frac{h}{10^{2j}} + \frac{h}{10^{3j}} + \cdots$$

where the block h has j digits. This is a geometric series with sum $r = h/(10^j - 1)$, which is a rational number. Finally, suppose that g and h are nonempty. By our observations above, there are positive integers j and k such that r can be written in the form

$$r = 0.g\bar{h} = 0.g + 0.00\cdots 0\bar{h}\cdots = 0.g + \frac{0.\bar{h}}{10^k} = \frac{g}{10^k} + \frac{h}{10^k\,(10^j - 1)}$$

We conclude that if r has a decimal expansion that is repeating, then r is a rational number.

Now we assume that r is a rational number in $(0, 1)$, say $r = p/q$, in reduced form. We will show that r has a repeating decimal expansion. The actual division of q into p nets a decimal expansion. In this decimal expansion there can be at most $q - 1$ distinct nonzero remainders before the remainder either returns to a previous nonzero integer remainder or becomes 0. Either way a repeating decimal expansion arises.

   b. Every rational number can be expressed as p/q, in reduced form. Let

$$G\left(\frac{p}{q}\right) = \begin{cases} 2^{p+1}3^q & \text{if } p \geq 1 \\ 0 & \text{if } p = 0 \\ 2^{-p+1}3^q + 1 & \text{if } p \leq -1 \end{cases}$$

The proof that G is one-to-one from the set of rational numbers onto the set of positive integers is similar to the proof appearing in Theorem 2.19.

3. In Exercise 2.2.3 we showed that the rationals are dense in the set of reals. To show that the irrationals are also dense in the reals, let U be any open interval of length d. Then there exists an integer n such that $\sqrt{2}/n < d$. Because U has length d and $\sqrt{2}(k+1)/n - \sqrt{2}k/n = \sqrt{2}/n < d$, it follows that for some integer k, $\sqrt{2}k/n$ is in U. Moreover, $\sqrt{2}k/n$ is irrational. Thus the irrationals are dense in the reals.

Now we will show that between any two rationals there are infinitely many irrationals, and vice versa. To that end, let A be a dense subset of the reals, and let a and b be arbitrary numbers with $a < b$. From (a, b) select an infinite number of disjoint open subintervals $A_1, A_2, A_3, \ldots$. One way is to let

$$A_k = \left\{\left(\frac{1}{k}a + \frac{k-1}{k}b, \frac{1}{k+1}a + \frac{k}{k+1}b\right)\right\}_{k=1}^{\infty}$$

Since A is dense in the reals, it follows that A_k contains a number in A. Consequently there are infinitely many members of A in (a, b). The desired result follows if we either let A denote the irrationals and a and b any two rationals, or let A denote the rationals and a and b any two irrationals.

4. a. Let E be countable, and assume that $D \subseteq E$. List the elements of E as $e_1, e_2, e_3, \ldots$. Now we strike out of the list those elements not in D. What remains is a list of D, which means that D is countable.

   b. The contrapositive of the above statement is the statement that if D is uncountable and $D \subseteq E$, then so is E.

5. a. Let S denote the set in question. Then S contains .3, .03, .003, …, which converges to 0. But 0 is not in S, so S is not closed. To show that S is not open, notice that any non-empty open interval in $(0, 1)$ contains an interval I of the form $I = ((k-1)/10^n, k/10^n)$, where k and n are positive integers with $k < n$. Now I contains numbers whose decimal expansions have the form $x_1x_2 \cdots x_n 11 x_{n+3} \cdots$. Such a number necessarily has a 1 in its decimal expansion, so is not in S. Thus S is not open.

   b. By the solution of part (a), S is totally disconnected.

6. Let $B_n = [0, 1 + 1/n]$ for $n = 1, 2, \ldots$. The sequence is a nested collection of bounded closed intervals, the lengths of which converge to 1 as n increases without bound.

7. We find that $D_n(C) = \displaystyle\sum_{k=1}^{2^n} \frac{1}{3^n} = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$. Thus $\lim_{n\to\infty} D_n(C_{1/5}) = 0$. This is reasonable.

8. a. At the nth stage, 2^{n-1} intervals are removed, each of length $1/5^n$. Thus

$$D_n(C_{1/5}) = 1 - \sum_{k=1}^{n} \frac{2^{k-1}}{5^k} = 1 - \frac{1}{5}\sum_{k=0}^{n-1}\left(\frac{2}{5}\right)^k = 1 - \frac{1}{5}\,\frac{1-(1/5)^n}{1-2/5} = \frac{2}{3} + \frac{1}{3}\left(\frac{2}{5}\right)^n$$

   b. From part (a), $\lim_{n\to\infty} D_n(C_{1/5}) = 2/3$.

9. Let $\mu > 4$, and suppose that x is a periodic point of Q_μ. If x or any given iterate of x lies outside $[0, 1]$, then successive iterates of x diverge to $-\infty$, which is impossible since x is by hypothesis a periodic point. Thus x and all of its iterates lie in $[0, 1]$, which means that x is in C_μ.

10. Let $\mu = 5$. Then direct computation shows that $J_{11} = [0, \frac{1}{2} - \frac{\sqrt{5}}{10}]$ and $J_{12} = [\frac{1}{2} + \frac{\sqrt{5}}{10}, 1]$, so that

$$J_{21} = [0, \frac{1}{2} - \frac{\sqrt{15+2\sqrt{5}}}{10}] \quad \text{and} \quad J_{22} = [\frac{1}{2} - \frac{\sqrt{15-2\sqrt{5}}}{10}, \frac{1}{2} - \frac{\sqrt{5}}{10}]$$

It is easy to check that J_{22} is longer than J_{21}.

11. By solving $Q_\mu(x) = 1$ for x, we discover that $J_{11} = [0, (1 - \sqrt{(\mu-4)/\mu})/2]$. Because the graph of Q_μ is concave downward, it follows that the minimum value of Q'_μ on J_{11} occurs at the right-hand endpoint. At that point we obtain

$$Q'_\mu\left(\frac{1}{2} - \frac{1}{2}\sqrt{(\mu-4)/\mu}\right) = \mu - 2\mu\left(\frac{1}{2} - \frac{1}{2}\sqrt{(\mu-4)/\mu}\right) = \sqrt{\mu(\mu-4)}$$

Since $\sqrt{\mu(\mu-4)} = 1$ if and only if $\mu^2 - 4\mu - 1 = 0$, which occurs if and only if $\mu = 2 \pm \sqrt{5}$, we deduce that if $\mu > 2 + \sqrt{5}$, then $Q'_\mu(x) > 1$ for all x in J_{11}. Since the graph of Q_μ is symmetric about the line $x = 1/2$, we conclude that if $\mu > 2 + \sqrt{5}$, then $|Q'_\mu(x)| > 1$ for all x in J_{12}.

12. Let x and y be distinct numbers in $[0, 1]$, and assume that y is not in C_μ. Since C_μ is totally disconnected, we can find such a y as close to x as we wish. If x is in C_μ, then since the iterates of x remain in $[0, 1]$ and the iterates of y diverge to $-\infty$, it follows that for some n, $|Q_\mu^{[n]}(x) - Q_\mu^{[n]}(y)| \geq 1$. If x is not in C_μ, then the iterates of x also diverge to $-\infty$, so we can find an n so large that $Q_\mu^{[n]}(x) < 0$ and $Q_\mu^{[n]}(y) < 0$. Since $Q'_\mu > \mu > 1$ on $(-\infty, 0)$, it follows that $|Q_\mu^{[n+1]}(x) - Q_\mu^{[n+1]}(y)| \geq \mu|x - y|$. Inductively we find that $|Q_\mu^{[n+k]}(x) - Q_\mu^{[n+k]}(y)| \geq \mu^k|x - y|$, for $k = 1, 2, \ldots$. Since $\lim_{\mu\to\infty} \mu^k = \infty$, we conclude that the iterates of x and y eventually separate by more than 1. Consequently Q_μ has sensitive dependence on $[0, 1]$.

13. If $H_\mu(x) = H_\mu(y)$, then x and y lie in the same subinterval J_{nk} for all n. Since the lengths of those intervals approach 0, this implies that $x = y$. To see that H_μ is onto, let $z_1z_2z_3\cdots$ be an element of A. Pick intervals $J_{1k(1)}, J_{2k(2)}, J_{3k(3)}, \ldots$, such that $k(1) = z_1$, and such that $J_{(n+1)k(n+1)} \subset J_{nk(n)}$, and such that $k(n + 1)$ is equivalent to z_{n+1} modulo 2. Then there exists a unique number x in the intersection of the $J_{nk(n)}$'s. That number x has the property that $H_\mu(x) = z_1z_2z_3\cdots$. Therefore H_μ is onto.

14. Suppose that $\|X - Y\| < 1/2^n$ but that there exists a $k \leq n$ such that $x_k \neq y_k$. Then

$$\|X - Y\| = \sum_{j=1,\, j\neq k}^{\infty} \frac{|x_j - y_j|}{2^j} + \frac{1}{2^k} \geq \frac{1}{2^n}$$

This contradicts the assumption. Therefore at least the first n elements of the sequences for X and Y coincide.

15. Denote by $d(x, y)$ the metric in D.

   a. Suppose that $f: D \to D$. If x is in D, then we say that f has sensitive dependence on initial conditions at x if there exists an $\varepsilon > 0$ such that for each $\delta > 0$ there exists a y in D and a positive integer n such that $d(x, y) < \delta$ and $d(f^{[n]}(x), f^{[n]}(y)) > \varepsilon$. If f has sensitive dependence on initial conditions at each x in D, then we say that f has sensitive dependence on initial conditions on D.

   b. A set A is called dense in D if for any x in D and any $\varepsilon > 0$, there is an a in A such that $d(a, x) < \varepsilon$. A dense set of periodic points is then defined in the obvious way.

   c. Using the solution of part (b), one defines a dense orbit in the obvious way.