CHAPTER 4

SECTION 4.1

1. Since $\det\begin{pmatrix} -2-\lambda & 9 \\ 1 & -2-\lambda \end{pmatrix} = \lambda^2+4\lambda-5 = (\lambda+5)(\lambda-1)$, the eigenvalues are -5 and 1. Therefore $\mathbf{0}$ is a saddle point. To find an eigenvector for the eigenvalue -5, we notice that

$$(-5)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 9 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x+9y \\ x-2y \end{pmatrix}$$

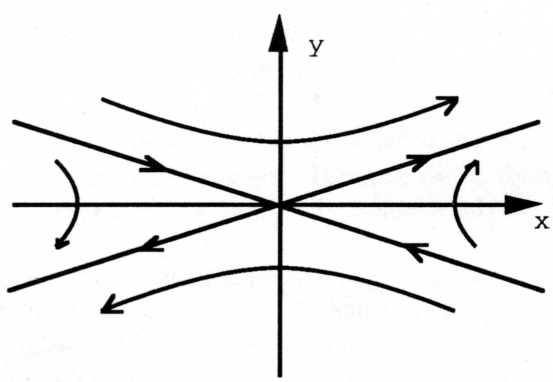
Thus $-5x = -2x+9y$, so that $x = -3y$. Therefore $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an eigenvector. To find an eigenvector for the eigenvalue 1, we notice that

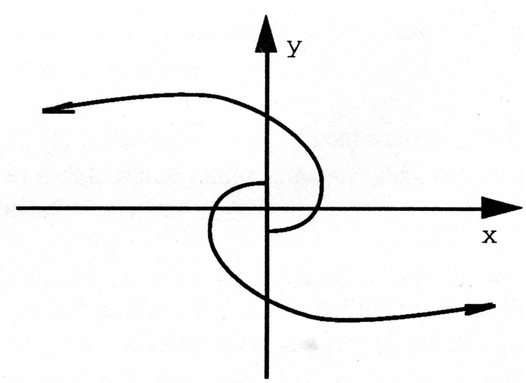
$$(1)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 9 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x+9y \\ x-2y \end{pmatrix}$$

It follows that $x = -2x+9$, so that $x = 3y$. Therefore $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector. The general solution is

$$X(t) = p\begin{pmatrix} 3 \\ 1 \end{pmatrix}e^t + q\begin{pmatrix} -3 \\ 1 \end{pmatrix}e^{-5t}$$

The portrait appears next, to the left.





2. Since $\det\begin{pmatrix} 2-\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix} = \lambda^2-5\lambda+8$, the eigenvalues are $(5 \pm i\sqrt{7})/2$. Thus $\mathbf{0}$ is an unstable spiral point. The portrait appears above, to the right.

3. Since $\det\begin{pmatrix} -2-\lambda & -1 \\ 0 & -2-\lambda \end{pmatrix} = (\lambda+2)^2$, the single eigenvalue is -2, so that $\mathbf{0}$ is an asymptotically stable degenerate node. To find an eigenvector for the eigenvalue -2, we notice that

$$(-2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x-y \\ -2y \end{pmatrix}$$

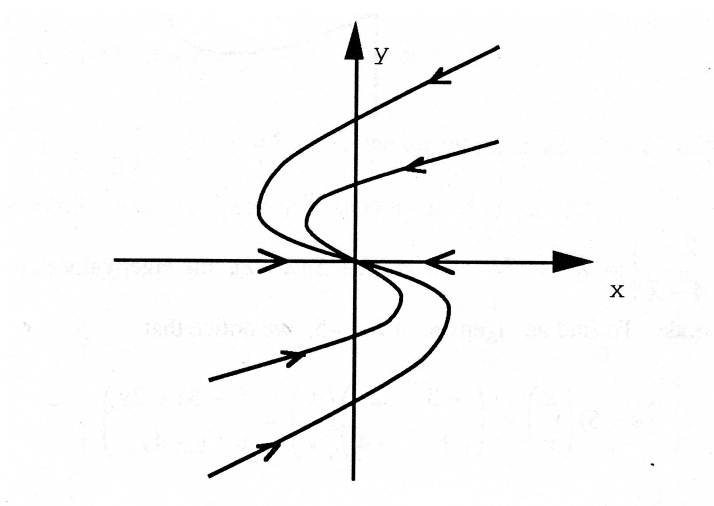
Therefore $-2x = -2x - y$, so that $y = 0$. Thus $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector. Next, as in the solution of Example 3, we need to find v and w such that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2-(-2) & -1 \\ 0 & -2-(-2) \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -w \\ 0 \end{pmatrix}$$

Consequently $w = -1$, and we can let $v = 0$. It follows that the general solution is given by

$$X(t) = p\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + q\begin{pmatrix} t \\ -1 \end{pmatrix} e^{-2t}$$

The portrait appears next.



4. Since $\det\begin{pmatrix} 1-\lambda & 5 \\ -1 & -1-\lambda \end{pmatrix} = \lambda^2 + 4$, the eigenvalues are $2i$ and $-2i$. Thus $\mathbf{0}$ is a stable center. To find an eigenvector, we notice that

$$(2i)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+5y \\ -x-y \end{pmatrix}$$

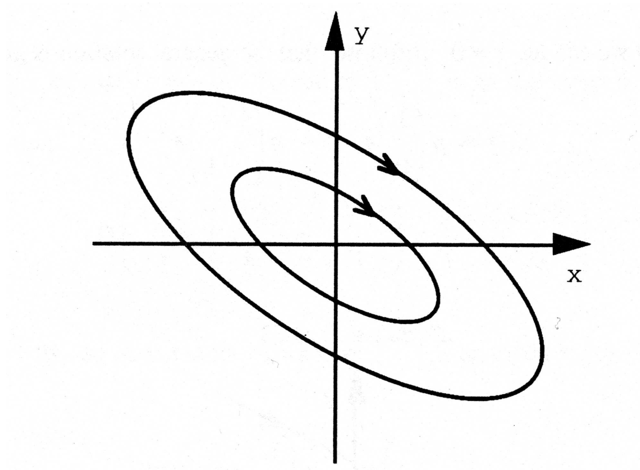
Therefore $2ix = x + 5y$, so that $y = (-1+2i)x/5$. If $x = 5$, then $y = -1 + 2i$. Thus

$$\begin{pmatrix} 5 \\ -1+2i \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} i$$

is an eigenvalue for $2i$. By (15) we conclude that the general solution is given by

$$X(t) = p\left(\begin{pmatrix} 5 \\ -1 \end{pmatrix}\cos 2t - \begin{pmatrix} 5 \\ 2 \end{pmatrix}\sin 2t\right) + q\left(\begin{pmatrix} 5 \\ -1 \end{pmatrix}\sin 2t + \begin{pmatrix} 5 \\ 2 \end{pmatrix}\cos 2t\right)$$

The portrait appears next.



5. Since $\det\begin{pmatrix} -3-\lambda & 2 \\ 1 & -4-\lambda \end{pmatrix} = \lambda^2 + 7\lambda + 10 = (\lambda+5)(\lambda+2)$, the eigenvalues are -5 and -2. Thus $\mathbf{0}$ is an asymptotically stable node. To find an eigenvector for -5, we notice that

$$(-5)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3x+2y \\ x-4y \end{pmatrix}$$

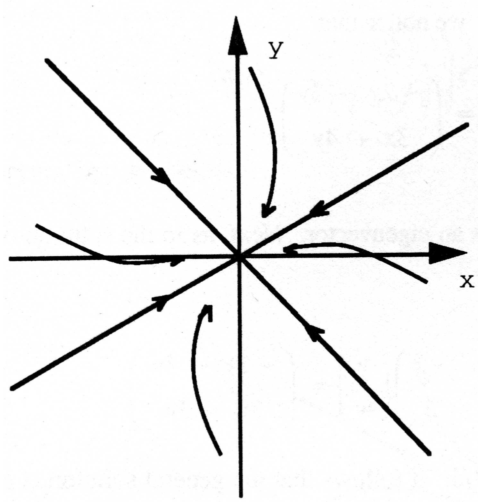
Thus $-5x = -3x + 2y$, so that $y = -x$. Therefore $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector. To find an eigenvector for the eigenvalue -2, we notice that

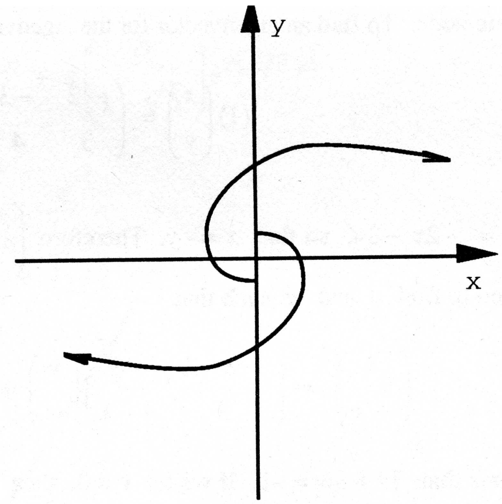
$$(-2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3x+2y \\ x-4y \end{pmatrix}$$

Thus $-2x = -3x + 2y$, so that $x = 2y$. Therefore $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector. The general solution is

$$X(t) = p\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{-2t} + q\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-5t}$$

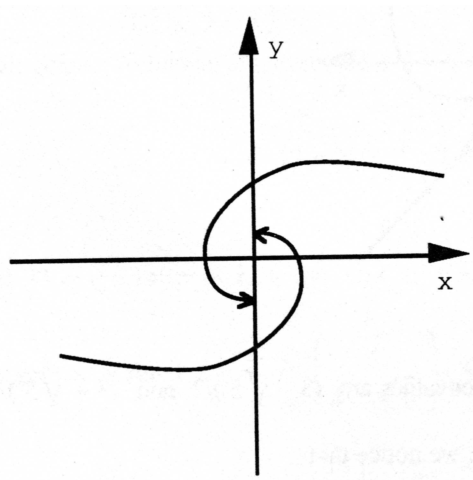
The portrait is next, to the left.

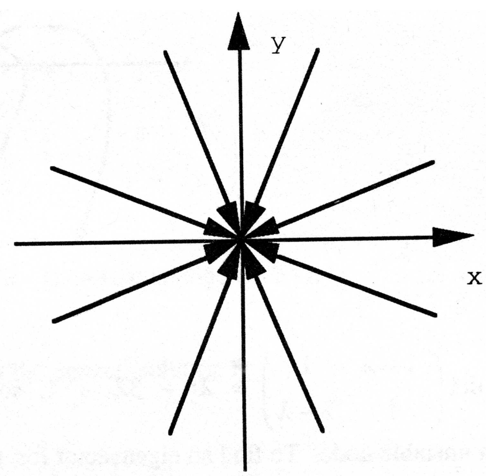




6. Since $\det\begin{pmatrix} 1-\lambda & 1 \\ -9 & 3-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 12$, the eigenvalues are $2 - 2\sqrt{2}i$ and $2 + 2\sqrt{2}i$. Therefore $\mathbf{0}$ is an unstable spiral point. The portrait appears above, to the right.

7. Since $\det\begin{pmatrix} -2-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix} = \lambda^2 + 4\lambda + 8$, the eigenvalues are $-2 - 2i$ and $-2 + 2i$. Therefore $\mathbf{0}$ is an asymptotically stable spiral point. The portrait is below, to the left.





8. Since $\det\begin{pmatrix} -3-\lambda & 0 \\ 0 & -3-\lambda \end{pmatrix} = (\lambda+3)^2$, the lone eigenvalue is -3. Thus $\mathbf{0}$ is an asymptotically stable degenerate node; more specifically, it is a star solution. The portrait appears above, to the right.

9. Since $\det\begin{pmatrix} -2-\lambda & -3 \\ 3 & 4-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2$, the lone eigenvalue is 1. Thus $\mathbf{0}$ is an unstable degenerate node. To find an eigenvector for the eigenvalue 1, we notice that

$$(1)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x-3y \\ 3x+4y \end{pmatrix}$$

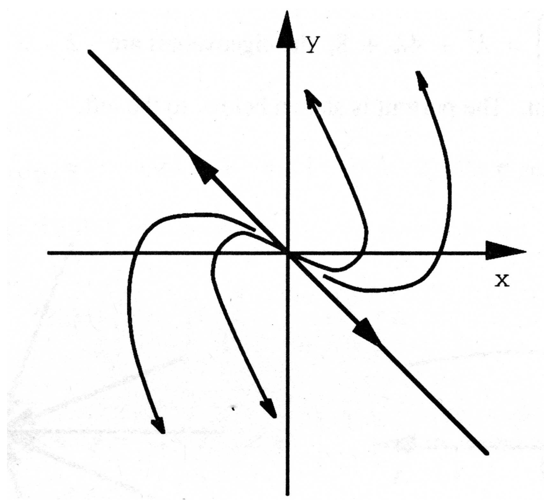
Thus $x = -2x - 3y$, so that $x = -y$. Therefore $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector. Next, as in the solution of Example 3, we need to find v and w such that

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2-1 & -3 \\ 3 & 4-1 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -3v-3w \\ 3v+3w \end{pmatrix}$$

This means that $3v + 3w = -1$. If we let $v = 0$, then $w = -1/3$. It follows that the general solution is given by

$$X(t) = p\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + q\begin{pmatrix} t \\ -1/3-t \end{pmatrix} e^t$$

The portrait appears next.



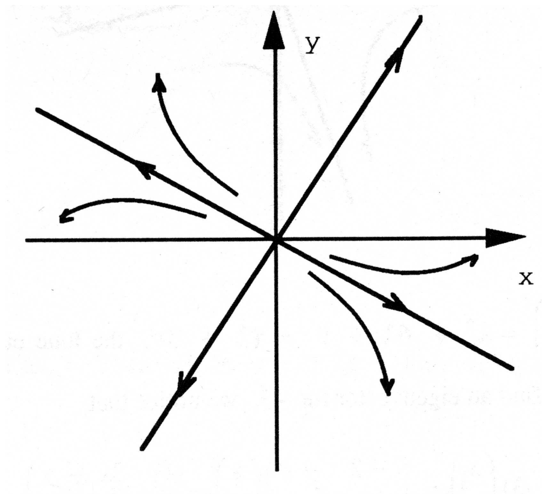
10. Since $\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 1$, the eigenvalues are $(3-\sqrt{5})/2$ and $(3+\sqrt{5})2$. Thus $\mathbf{0}$ is an unstable node. To find an eigenvector for $(3-\sqrt{5})/2$, we notice that

$$\frac{1}{2}(3-\sqrt{5})\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+2y \end{pmatrix}$$

Thus $(3-\sqrt{5})x/2 = x + y$, so that $y = (1-\sqrt{5})x/2$. Therefore $\begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix}$ is an eigenvector. Similarly, $\begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}$ is an eigenvector for $(3+\sqrt{5})/2$. The general solution is given by

$$X(t) = p\begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix} e^{(3+\sqrt{5})t/2} + q\begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix} e^{(3-\sqrt{5})t/2}$$

The portrait appears next.



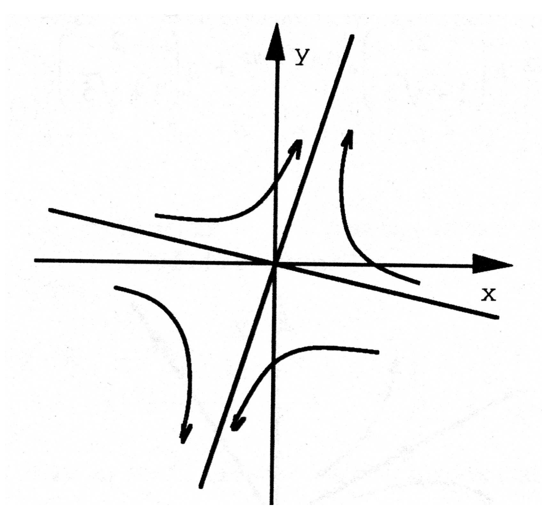
11. Since $\det \begin{pmatrix} -1-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix} = \lambda^2 - 2\lambda - 5$, the eigenvalues are $1-\sqrt{6}$ and $1+\sqrt{6}$. Therefore $\mathbf{0}$ is a saddle point. To find an eigenvector for $1-\sqrt{6}$, we notice that

$$(1-\sqrt{6})\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x+2y \\ x+3y \end{pmatrix}$$

Thus $(1-\sqrt{6})x = -x+2y$, so that $y = (2-\sqrt{6})x/2$. Therefore an eigenvalue is $\begin{pmatrix} 2 \\ 2-\sqrt{6} \end{pmatrix}$. Similarly, an eigenvector for $1+\sqrt{6}$ is $\begin{pmatrix} 2 \\ 2+\sqrt{6} \end{pmatrix}$. Consequently the general solution is

$$X(t) = p\begin{pmatrix} 2 \\ 2+\sqrt{6} \end{pmatrix} e^{(1+\sqrt{6})t} + q\begin{pmatrix} 2 \\ 2-\sqrt{6} \end{pmatrix} e^{(1-\sqrt{6})t}$$

The portrait appears next.



12. Since $\det\begin{pmatrix} -2-\lambda & -1 \\ 1 & -4-\lambda \end{pmatrix} = \lambda^2+6\lambda+9 = (\lambda+3)^2$, the lone eigenvalue is -3. Thus $\mathbf{0}$ is an asymptotically stable node. To find an eigenvector for -3, we notice that

$$(-3)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x-y \\ 1-4y \end{pmatrix}$$

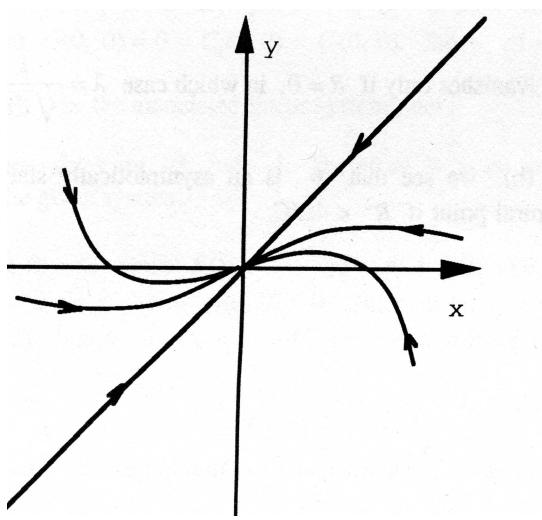
Thus $-3x = -2x - y$, so that $y = x$. Therefore $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector. Next, as in the solution of Example 3, we need to find v and w such that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2+3 & -1 \\ 1 & -4+3 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v-w \\ v-w \end{pmatrix}$$

This means that $1 = v - w$, so that $v = 1 + w$. As a result, we can let $w = 1$, so that $v = 2$. By (14) the general solution is given by

$$X(t) = p\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-3t} + q\begin{pmatrix} 2+t \\ 1+t \end{pmatrix}e^{-3t}$$

The portrait appears next.



13. Suppose that $X_2(t) = cX_1(t)$ for all t. Then $dX_2/dt = c\,dX_1/dt$. However, $dX_1/dt = \lambda X_1$, and $dX_2/dt = \mu X_2$. Thus

$$\mu X_2 = \frac{dX_2}{dt} = c\frac{dX_1}{dt} = c\lambda X_1 = \lambda X_2$$

We conclude that $\mu = \lambda$.

14. (a) Recall that $\mathbf{0}$ is a center only if the eigenvalues are pure imaginary. Since

$$\det\begin{pmatrix} a-\lambda & 10 \\ 1 & 3-\lambda \end{pmatrix} = (a-\lambda)(3-\lambda) - 10 = \lambda^2 - (a+3)\lambda + 3a - 10$$

the roots of this polynomial are

$$\frac{1}{2}\left(a+3 \pm \sqrt{(a+3)^2 - 12a + 40}\right) = \frac{1}{2}\left(a+3 \pm \sqrt{(a-3)^2 + 40}\right)$$

But the number inside the square root is always positive, so that the roots of the polynomial are never imaginary. Thus there is no value of a such that $\mathbf{0}$ is a center.

Note: If $dx/dt = ax - 10y$ in the given system, then a similar calculation would yield

$$\frac{1}{2}\left(a+3 \pm \sqrt{(a-3)^2 - 40}\right)$$

which would be pure imaginary if $a = -3$, so that $\mathbf{0}$ would be a center if $a = -3$.

(b) Since there is no such value of a, uniqueness is not relevant.

(c) Again, there is no possibility of $\mathbf{0}$ being a spiral, since the roots of the polynomial in (a) are always real.

Note: If $dx/dt = ax - 10y$ in the given system, then if $a \neq -3$ and $(a-3)^2 < 40$, it would follow that the roots of the polynomial would have imaginary parts. In other words, $\mathbf{0}$ would be a spiral if $3 - 2\sqrt{10} < a < 3 + 2\sqrt{10}$ and $a \neq 3$.

15. (a) The required system is

$$\begin{aligned} \frac{dQ}{dt} &= I \\ \frac{dI}{dt} &= -\frac{1}{LC}Q - \frac{R}{L}I \end{aligned}$$

(b) First we calculate that

$$\det\begin{pmatrix} 0-\lambda & 1 \\ -1/LC & -R/L-\lambda \end{pmatrix} = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}$$

which equals 0 if

$$\lambda = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

Thus the real part of λ vanishes only if $R = 0$, in which case $\lambda = \dfrac{1}{\sqrt{LC}}i$. The period is $2\pi\sqrt{LC}$.

(c) From the formula in (b), we see that $\mathbf{0}$ is an asymptotically stable node if $R^2 > 4L/C$, and is an asymptotically stable spiral point if $R^2 < 4L/C$.

SECTION 4.2

1. Let $F(x, y) = \ln(1 + y^2)$. Then $F(0, 0) = 0$. Since $F_y(x, y) = 2y/(1 + y^2)$, it follows that $F_x(0, 0) = 0 = F_y(0, 0)$. Next, let $G(x, y) = -x^2y$. Then $G(0, 0) = 0 = G_x(0, 0) = G_y(0, 0)$. Since $ad - bc = 3 \neq 0$, it follows that the given system is almost linear at $\mathbf{0}$. For the associated linear system, $\det\begin{pmatrix} -1-\lambda & 0 \\ 2 & -3-\lambda \end{pmatrix} = \lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1)$, so that the eigenvalues are -3 and -1. Therefore $\mathbf{0}$ is an asymptotically stable node of both the associated linear system and the given system.

2. Let $F(x, y) = 1 - e^{xy}$. Then $F(0, 0) = 0$. Since $F_x(x, y) = -ye^{xy}$ and $F_y(x, y) = -xe^{xy}$, it follows that $F_x(0, 0) = 0 = F_y(0, 0)$. Next, let $G(x, y) = x \sin y$. Then $G(0, 0) = 0$. Since $G_x(x, y) = \sin y$ and $G_y(x, y) = x \cos y$, it follows that $G_x(0, 0) = 0 = G_y(0, 0)$. Since $ad - bc = -1 \neq 0$, we find that the given system is almost linear at $\mathbf{0}$. For the associated linear system, $\det\begin{pmatrix} -1-\lambda & 2 \\ 1 & -1-\lambda \end{pmatrix} = \lambda^2 + 2\lambda - 1$, so that the eigenvalues are $-1 - \sqrt{2}$ and $-1 + \sqrt{2}$. Thus $\mathbf{0}$ is an unstable saddle point of both the associated linear system and the given system.

3. We notice that $dx/dt = 0$ if $y = 0$, and $dy/dt = 0$ if $x - x^3 = 0$, so that $x = 0, -1$, or 1. Thus the critical points of the system are $(0, 0), (-1, 0)$, and $(1, 0)$. For the critical point $(0, 0)$, let $F(x, y) = 0$ and $G(x, y) = x^3$. Therefore F and G and their first partial derivatives are 0 at $(0, 0)$. Since $ad - bc = -1 \neq 0$, it follows that the system is almost linear at $(0, 0)$. Since $\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1$, the eigenvalues of the associated linear system are -1 and 1. Consequently $(0, 0)$ is an unstable saddle point of both the associated linear system and the given system. For the critical point $(-1, 0)$, let $x = -1 + u$ and $y = v$. Then the given system becomes

$$\begin{aligned} \frac{du}{dt} &= v \\ \frac{dv}{dt} &= (u-1) - (u-1)^3 = -2u + 3u^2 - u^3 \end{aligned}$$

Thus we let $F(u, v) = 0$ and $G(u, v) = 3u^2 - u^3$. Then F and G and their first partial derivatives are 0 at $(0, 0)$. Since $ad - bc = 2 \neq 0$, the given system is almost linear at $(-1, 0)$. Since $\det\begin{pmatrix} -\lambda & 1 \\ -2 & \lambda \end{pmatrix} = \lambda^2 + 2$, the eigenvalues are $-\sqrt{2}i$ and $\sqrt{2}i$. Therefore we cannot conclude anything about the nature of the critical point. Finally, for the critical point $(1, 0)$, let $x = 1 + u$ and $y = v$. Then the given system becomes

$$\begin{aligned} \frac{du}{dt} &= v \\ \frac{dv}{dt} &= (u+1) - (u+1)^3 = -2u - 3u^2 - u^3 \end{aligned}$$

We obtain the same results for $(1, 0)$ as for $(-1, 0)$: the system is almost linear at $(1, 0)$, and we cannot conclude anything about the nature of the critical point.

4. We notice that $dx/dt = 0$ if $y = 0$, and $dy/dt = 0$ if $-x + x^3 = 0$, so that $x = 0, -1$, or 1. Thus the critical points are $(0,0), (-1,0)$, and $(1,0)$. For the critical point $(0,0)$ we have $F(x,y) = 0$ and $G(x,y) = -x + x^3$. Then F and G and their first partial derivatives are 0 at $(0,0)$. Since $ad - bc = 1 \neq 0$, it follows that the given system is almost linear at $(0,0)$. Since $\det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1$, the eigenvalues of the associated matrix are i and $-i$. Therefore we cannot conclude anything about the nature of the critical point $(0,0)$. For the critical point $(-1,0)$, let $x = -1 + u$ and $y = v$. Then the given system becomes

$$\begin{aligned} \frac{du}{dt} &= v \\ \frac{dv}{dt} &= -(u-1) + (u-1)^3 = 2u - 3u^2 + u^3 \end{aligned}$$

Thus we let $F(u,v) = 0$ and $G(u,v) = -3u^2 + u^3$. Then F and G and their first partial derivatives are 0 at $(0,0)$. Since $ad - bc = -2 \neq 0$, the given system is almost linear at $(-1,0)$. Since $\det \begin{pmatrix} -\lambda & 1 \\ 2 & -\lambda \end{pmatrix} = \lambda^2 - 2$, the eigenvalues are $-\sqrt{2}$ and $\sqrt{2}$. Therefore $(-1,0)$ is an unstable saddle point of both the associated linear system and the given system. Finally, for the critical point $(1,0)$, let $x = 1 + u$ and $y = v$. Then the given system becomes

$$\begin{aligned} \frac{du}{dt} &= v \\ \frac{dv}{dt} &= -(u+1) + (u+1)^3 = 2u + 3u^2 + u^3 \end{aligned}$$

We obtain the same results for $(1,0)$ as for $(-1,0)$: the system is almost linear at $(1,0)$, and $(1,0)$ is an unstable saddle point of both the associate linear system and the given system.

5. We notice that $dx/dt = 2x - x^2 - xy$ and $dy/dt = -y + xy$. In order for $dx/dt = 0 = dy/dt$, we must have $2x - x^2 - xy = 0$ and $-y + xy = 0$. If $-y + xy = 0$, then $y = 0$ or $x = 1$. If $y = 0$, then $2x - x^2 - xy = 0$ implies that $x = 0$ or $x = 2$. If $y \neq 0$, then since $x = 1$ and since $2x - x^2 - xy = 0$, we find that $y = 1$. Thus the critical points are $(0,0), (1,1)$, and $(2,0)$. For the critical point $(0,0)$, let $F(x,y) = -x^2 - xy$ and $G(x,y) = xy$. Since F and G and their first partial derivatives are 0 at $(0,0)$, and since $ad - bc = -2 \neq 0$, we know that the system is almost linear at $(0,0)$. Since $\det \begin{pmatrix} 2-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} = (\lambda+1)(\lambda-2)$, we find that the eigenvalues of the associated linear system are -1 and 2. Therefore $(0,0)$ is an unstable saddle point of both the associated linear system and the given system.

   For the critical point $(1,1)$, let $x = 1 + u$ and $y = 1 + v$. Then the given system becomes

$$\begin{aligned} \frac{du}{dt} &= 2(1+u) - (1+u)^2 - (1+u)(1+v) = -u - v - u^2 - uv \\ \frac{dv}{dt} &= -(1+v) + (1+u)(1+v) = u + uv \end{aligned}$$

Thus we let $F(u,v) = -u^2 - uv$ and $G(u,v) = uv$. Then F and G and their first partial derivatives are 0 at $(0,0)$. Since $ad - bc = 1 \neq 0$, the given system is almost linear at $(1,1)$. Since $\det\begin{pmatrix} -1-\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + \lambda + 1$, the eigenvalues of the associated system are $(-1-\sqrt{3}i)/2$ and $(-1+\sqrt{3}i)/2$. Therefore $(1,1)$ is an asymptotically stable spiral point for the associated linear system and for the given system.

For the critical point $(2,0)$, let $x = 2 + u$ and $y = v$. Then the given system becomes

$$\begin{aligned} \frac{du}{dt} &= 2(2+u) - (2+u)^2 - (2+u)v = -2u - 2v - u^2 - uv \\ \frac{dv}{dt} &= -v + (2+u)v = v + uv \end{aligned}$$

Thus we let $F(u,v) = -u^2 - uv$ and $G(u,v) = uv$. Then F and G and their first partial derivatives are 0 at $(0,0)$. Since $ad - bc = -2 \neq 0$, the given system is almost linear at $(2,0)$. Since $\det\begin{pmatrix} -2-\lambda & -2 \\ 0 & 1-\lambda \end{pmatrix} = (\lambda+2)(\lambda-1)$, the eigenvalues are -2 and 1. Therefore $(2,0)$ is an unstable saddle point for the associated linear system and the given system.

6. We notice that $dx/dt = 0$ if $y - x^3 = 0$, so that $y = x^3$. Since $dy/dt = 0$ if $1 - xy = 0$, if we substitute x^3 for y in the equation $1 - xy = 0$, we find that $1 - x(x^3) = 0$, so that $x = -1$ or $x = 1$. Using the fact that $y = x^3$, we conclude that the critical points are $(-1,-1)$ and $(1,1)$. For the critical point $(-1,-1)$, let $x = -1 + u$ and $y = -1 + v$. Then the given system becomes

$$\begin{aligned} \frac{du}{dt} &= (-1+v) - (-1+u)^3 = -3u + v + 3u^2 - u^3 \\ \frac{dv}{dt} &= 1 - (-1+u)(-1+v) = u + v - uv \end{aligned}$$

Thus we let $F(u,v) = 3u^2 - u^3$ and $G(u,v) = -uv$. Then F and G and their first partial derivatives are 0 at $(0,0)$. Since $ad - bc = -4 \neq 0$, the given system is almost linear at $(-1,-1)$. Since $\det\begin{pmatrix} -3-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = \lambda^2 + 2\lambda - 4$, the eigenvalues are $-1-\sqrt{5}$ and $-1+\sqrt{5}$. As a result, $(-1,-1)$ is an unstable saddle point for the associated linear system and for the given system.

For the critical point $(1,1)$, let $x = 1 + u$ and $y = 1 + v$. Then the given system becomes

$$\begin{aligned} \frac{du}{dt} &= (1+v) - (1+u)^3 = -3u + v - 3u^2 - u^3 \\ \frac{dv}{dt} &= 1 - (1+u)(1+v) = -u - v - uv \end{aligned}$$

Thus we let $F(u,v) = -3u^2 - u^3$ and $G(u,v) = -uv$. Then F and G and their first derivatives are 0 at $(0,0)$. Since $ad - bc = 4 \neq 0$, the given system is almost linear at $(1,1)$. Since $\det\begin{pmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = \lambda^2 + 4\lambda + 4$, the associated linear system has a unique eigenvalue, -2.

As a result, $(1,1)$ is an asymptotically stable degenerate node of the associated linear system. Although $(1,1)$ is therefore asymptotically stable for the given system, we cannot conclude from this information whether it is a node or not.

7. (a) Let $F(x,y) = -x(x^2+y^2)$ and $G(x,y) = -y(x^2+y^2)$. Since F and G and all their first partial derivatives are 0 at $(0,0)$, and since $ad - bc = 1 \neq 0$, we know that the system is almost linear at $\mathbf{0} = (0,0)$.

   (b) Note that the associated linear system is

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x \end{aligned}$$

   Since $\det \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 + 1$, the eigenvalues of the associated linear system are $-i$ and i. Thus $\mathbf{0}$ is a stable center for the associated linear system.

   (c) Let $x = r\cos\theta$ and $y = r\sin\theta$, and use (5) and (6) to convert the system to polar coordinates. We obtain

$$\begin{aligned} \frac{dr}{dt} &= -r^3 \\ \frac{d\theta}{dt} &= -1 \end{aligned}$$

   We can solve these equations (separately), obtaining

$$r^2 = \frac{1}{2(t+a)} \quad \text{and} \quad \theta = -t + b$$

   for arbitrary constants a and b. Therefore the trajectory of any point spirals clockwise toward the origin as t increases without bound. Consequently $\mathbf{0}$ is asymptotically stable.

8. (a) The nonlinear terms of the given system are negatives of the corresponding terms in the system of Exercise 7, so in a similar way the present system is almost linear at $\mathbf{0} = (0,0)$.

   (b) The given system has the same associated linear system as that in Exercise 7, so by the same calculations, $\mathbf{0}$ is a stable center of the associated linear system.

   (c) When we convert to polar coordinates by means of the equations $x = r\cos\theta$ and $y = r\sin\theta$, we obtain

$$\begin{aligned} \frac{dr}{dt} &= r^3 \\ \frac{d\theta}{dt} &= -1 \end{aligned}$$

   We can solve these equations (separately), obtaining

$$r^2 = \frac{-1}{2(t+a)} \quad \text{and} \quad \theta = -t + b$$

Since $r^2 \geq 0$, it follows that $t + a < 0$, so that $t < -a$. Thus r^2 increases without bound as t approaches $-a$ from the left. Consequently the trajectories all spiral clockwise to ∞ as t increases and approaches $-a$.

(d) Since all trajectories spiral toward ∞ by the result of (c), there cannot be any attractor.

9. (a) If we let $dx/dt = y$, then the given equation can be transformed into

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x + \epsilon y - \epsilon x^2 y \end{aligned}$$

Letting $F(x,y) = 0$ and $G(x,y) = -\epsilon x^2 y$, we see that this system is almost linear at $(0,0)$.

(b) Since $\det \begin{pmatrix} -\lambda & 1 \\ -1 & \epsilon - \lambda \end{pmatrix} = \lambda^2 - \epsilon\lambda + 1$, the eigenvalues of the system appearing in (a) are $(\epsilon - \sqrt{\epsilon^2 - 4})/2$ and $(\epsilon + \sqrt{\epsilon^2 - 4})/2$. If $0 < \epsilon < 2$, then the eigenvalues are complex, with positive real part. In that case, $\mathbf{0}$ is an unstable spiral point of both the auxiliary system and the nonlinear system. If $\epsilon = 2$, then there is a single positive real eigenvalue, ϵ. In that case, $\mathbf{0}$ is an unstable degenerate node of the auxiliary system, and hence is an unstable critical point of the nonlinear system. Finally, if $\epsilon > 2$, then there are two positive eigenvalues, so that $\mathbf{0}$ is an unstable node of both the auxiliary system and the nonlinear system. We conclude that in any case, $\mathbf{0}$ is unstable, and is a spiral point if $0 < \epsilon < 2$.

10. Letting $dx/dt = y$, we derive the corresponding system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= \frac{1}{2}x - ay - \frac{1}{2}x^3 \end{aligned}$$

Now $dx/dt = 0$ if $y = 0$. Therefore $dy/dt = 0$ if $x/2 - x^3/2 = 0$, which means that $x = 0, x = -1$, or $x = 1$. Thus the critical points are $(0,0), (-1,0)$, and $(1,0)$. For the critical point $(0,0)$, let $F(x,y) = 0$ and $G(x,y) = -x^3/2$. Since F and G and their first partial derivatives are 0 at $(0,0)$, and since $ad - bc = -1/2 \neq 0$, we know that the system is almost linear at $(0,0)$. Since $\det \begin{pmatrix} -\lambda & 1 \\ 1/2 & -a - \lambda \end{pmatrix} = \lambda^2 + a\lambda - 1/2$, we find that the eigenvalues of the associated linear system are $\left(-a - \sqrt{a^2 + 2}\right)/2$ and $\left(-a + \sqrt{a^2+2}\right)/2$. Since $\sqrt{a^2+2} > a$ and since $a > 0$ by hypothesis, we conclude that $(0,0)$ is an unstable saddle point of both the associated linear system and the given system.

For the critical point $(-1, 0)$, let $x = -1 + u$ and $y = v$. Then the given system becomes

$$\begin{aligned}\frac{du}{dt} &= v\\ \frac{dv}{dt} &= \frac{1}{2}(-1+u) - av - \frac{1}{2}(-1+u)^3 = -u - av + \frac{3}{2}u^2 - \frac{1}{2}u^3\end{aligned}$$

Thus we let $F(u, v) = 0$ and $G(u, v) = (3u^2 - u^3)/2$. Then F and G and their first partial derivatives are 0 at $(0, 0)$. Since $ad - bc = 1 \neq 0$, the given system is almost linear at $(-1, 0)$. Since $\det\begin{pmatrix} -\lambda & 1 \\ -1 & -a-\lambda \end{pmatrix} = \lambda^2 + a\lambda + 1$, the eigenvalues of the associated system are $\left(-a - \sqrt{a^2-4}\right)/2$ and $\left(-a + \sqrt{a^2-4}\right)/2$. If $0 < a < 2$, then $(-1, 0)$ is an asymptotically stable spiral point for both the associated linear system and the given system. If $a = 2$, then $(-1, 0)$ is asymptotically stable but we cannot conclude anything else about it. Finally, if $a > 2$, then $(-1, 0)$ is an asymptotically stable node of both the associated linear system and the given system.

For the critical point $(1, 0)$, let $x = 1 + u$ and $y = v$. Then the given system becomes

$$\begin{aligned}\frac{du}{dt} &= v\\ \frac{dv}{dt} &= \frac{1}{2}(1+u) - av - \frac{1}{2}(1+u)^3 = -u - av - \frac{3}{2}u^2 - \frac{1}{2}u^3\end{aligned}$$

The system is almost linear, and has precisely the same associated linear system as does the one analyzed for the critical point $(-1, 0)$. Therefore $(1, 0)$ has exactly the same features.

11. Letting $dx/dt = y$, we derive the corresponding system

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -ay + b\sin x\end{aligned}$$

Now $dx/dt = 0$ if $y = 0$. Therefore $dy/dt = 0$ if $b \sin x = 0$. Consequently the critical points of the system have the form $(n\pi, 0)$, where n is any integer. Let $x = n\pi + u$ and $y = v$.

If n is even, the the system is transformed into

$$\begin{aligned}\frac{du}{dt} &= v\\ \frac{dv}{dt} &= -av + b\sin(n\pi + u) = bu - av + (b\sin u - bu)\end{aligned}$$

If we let $F(u, v) = 0$ and $G(u, v) = b\sin u - bu$, then F and G and their first partial derivatives are 0 at $(0, 0)$, so that the uv-system is almost linear at $(0, 0)$. Since $\det\begin{pmatrix} -\lambda & 1 \\ b & -a-\lambda \end{pmatrix} = \lambda^2 + a\lambda - b$, the eigenvalues of the associated linear system are $\left(-a-\sqrt{a^2+4b}\right)/2$ and $\left(-a+\sqrt{a^2+4b}\right)/2$. Since $a > 0$ and $b > 0$ by hypothesis, it follows that $(0,0)$ is a saddle point of both the associated linear system and the uv-system, and hence of the given system at $(n\pi, 0)$.

If n is odd, then the system is transformed into

$$\begin{aligned}\frac{du}{dt} &= v \\ \frac{dv}{dt} &= -av + b\sin(n\pi + u) = -bu - av + (-b\sin u + bu)\end{aligned}$$

If we let $F(u,v) = 0$ and $G(u,v) = -b\sin u + bu$, then F and G and their first partial derivatives are 0 at $(0,0)$, so that the uv-system is almost linear at $(0,0)$. Since $\det\begin{pmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{pmatrix} = \lambda^2 + a\lambda + b$, the eigenvalues of the associated linear system are $\left(-a-\sqrt{a^2-4b}\right)/2$ and $\left(-a+\sqrt{a^2-4b}\right)/2$. Since $a > 0$ and $b > 0$ by hypothesis, it follows that $(0,0)$ is asymptotically stable for the uv-system, and is a node if $a^2 > 4b$ and is a spiral if $a^2 < 4b$. The same features hold for the given system at $(n\pi, 0)$. Finally, if $a^2 = 4b$, then $(0,0$ is a degenerate node, so we cannot conclude anything about the nature of the critical point of the given system at $(n\pi, 0)$.

12. Letting $dx/dt = y$, we derive the corresponding system

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -ay + x^3\end{aligned}$$

Now $dx/dt = 0$ if $y = 0$. Therefore $dy/dt = 0$ if $x^3 = 0$, so that $(0,0)$ is the only critical point of the system. If we let $F(x,y) = 0$ and $G(x,y) = x^3$, then F and G and their first partial derivatives are 0 at $(0,0)$. However, $ad - bc = 0$. Thus the system is not almost linear at $(0,0)$.

SECTION 4.3

1. Consider an elliptical orbit, as shown in Figure 4.12(a). Moreover, consider a point on the orbit where $x = 0$ and $dx/dt = y > 0$. At this point the pendulum is hanging straight down, and has positive angular velocity. Thus the pendulum must move to the right, which means that it must move to a position where x and y are both positive. Consequently the orbit is traversed clockwise.

2. The system in (4) is
$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -\frac{g}{L}\sin x - \frac{c}{mL}y\end{aligned}$$
Now if $dx/dt = 0$, then $y = 0$. Therefore if $dy/dt = 0$, then
$$0 = -\frac{g}{L}\sin x$$
so that $x = n\pi$ for some integer n. Thus $(0,0)$ is a critical point of (4). Next, we notice that the system can be rewritten as
$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -\frac{g}{L}x - \frac{c}{mL}y + (\frac{g}{L}x - \frac{g}{L}\sin x)\end{aligned}$$
If we let $F(x,y) = 0$ and $G(x,y) = gx/L - g(\sin x)/L$, then since
$$\frac{dG}{dt} = \frac{g}{L} - \frac{g}{L}\cos x \quad \text{and} \quad \frac{dG}{dy} = 0$$
it follows that F and G and their first partial derivatives are 0 at $(0,0)$. The system in (4) is thus almost linear at $(0,0)$.

3. The text shows that the eigenvalues are $(-c - \sqrt{c^2 - 4gmL})/2mL$ and $(-c + \sqrt{c^2 - 4gmL})/2mL$. The first has a negative real part because $c > 0$. Since m and L are positive, it follows that the real part of $\sqrt{c^2 - 4gmL}$ is less than c, so that the second eigenvalue has negative real part also.

4. The solution coincides with the final part of the solution of Exercise 4.2.11, where $a = -c/mL$ and $b = -g/L$.

5. (a) Multiplying both sides of (11) by $2d\theta/dt$ and dividing through by mL^2 yields
$$0 = 2\,\frac{d^2\theta}{dt^2}\,\frac{d\theta}{dt} + 2\,\frac{g}{L}\,(\sin\theta)\,\frac{d\theta}{dt} = \frac{d(d\theta/dt)^2}{dt} - \frac{2g}{L}\,\frac{d}{dt}(\cos\theta)$$
Now we integrate to obtain
$$c = \left(\frac{d\theta}{dt}\right)^2 - \frac{2g}{L}\,\cos\theta$$

Thus

$$\left(\frac{d\theta}{dt}\right)^2 = c + \frac{2g}{L}\cos\theta$$

(b) If $t = 0$, then by hypothesis, $\theta = 0$ and $d\theta/dt = \omega$. Then from the final equation in (a),

$$\omega^2 = c + \frac{2g}{L}\cos 0 = c + \frac{2g}{L}$$

Thus $c = \omega^2 - 2g/L$. Substituting for c in the final equation of (a) and using the half-angle formula $\sin^2(\theta/2) = (1 - \cos\theta)/2$, we deduce that

$$\left(\frac{d\theta}{dt}\right)^2 = \omega^2 - \frac{2g}{L} + \frac{2g}{L}\cos\theta = \omega^2 - \frac{2g}{L}(1 - \cos\theta) = \omega^2 - \frac{4g}{L}\sin^2\frac{\theta}{2}$$

Therefore

$$\frac{d\theta}{dt} = \sqrt{\omega^2 - \frac{4g}{L}\sin^2\frac{\theta}{2}} = \omega\sqrt{1 - \frac{4g}{\omega L}\sin^2\frac{\theta}{2}}$$

6. If $\omega > 2\sqrt{g/L}$, then $\omega^2 > 4g/L$. In this case the expression inside the square root in the final equation of the solution to Exercise 5 is positive, so that $d\theta/dt > 0$ for all t. Thus the pendulum rotates counterclockwise, with a minimum angular velocity of $\sqrt{\omega^2 - 4g/L}$ that occurs when $\theta = \pi$, that is, when the pendulum is in the upward vertical position.

7. if $\omega < 2\sqrt{g/L}$, then from the final equation of the solution to Exercise 5 there is an angle θ_0 between 0 and π such that $\omega^2 - \dfrac{4g}{L}\sin^2\dfrac{\theta_0}{2} = 0$, and thus

$$\theta_0 = 2\arcsin\left(\frac{\omega}{2}\sqrt{\frac{L}{g}}\right)$$

When $\theta = \theta_0$, we have $d\theta/dt = 0$, so the pendulum stops and begins to move in the reverse direction.

8. If $\omega = 2\sqrt{g/L}$, then the final equation of the solution to Exercise 5 becomes

$$\frac{d\theta}{dt} = 2\sqrt{g/L}\sqrt{1 - \sin^2\frac{\theta}{2}} = 2\sqrt{g/L}\cos\frac{\theta}{2}, \quad \text{where } -\pi/2 < \theta < \pi/2$$

If we separate variables, then we obtain

$$\frac{1}{2}\sec\frac{\theta}{2}\,d\theta = \sqrt{g/L}\,dt$$

Integration of both sides yields

$$\ln\left|\sec\frac{\theta}{2} + \tan\frac{\theta}{2}\right| = \sqrt{g/L}\,t + c$$

If $\theta(0) = 0$, then $c = 0$. Now if t increases without bound, then by the preceding equation, $\theta/2$ approaches $\pm\pi/2$, so that θ approaches $\pm\pi$. Thus the pendulum approaches the upward vertical position.

SECTION 4.4

1. Let $g(x) = (1+x)^3/x = x^2 + 3x + 3 + 1/x$ for $x > 0$. Then $f(a) = \pi^4 g(a^2)$. Now

$$g'(x) = 2x + 3 - \frac{1}{x^2} = \frac{2x^3 + 3x^2 - 1}{x^2} = \frac{2(x - 1/2)(x+1)^2}{x^2}$$

Since $\lim_{x\to 0^+} g(x) = \infty = \lim_{x\to\infty} g(x)$, and since $g'(x) = 0$ for $x > 0$ only when $x = 1/2$, it follows that $g(1/2)$ is the minimum value of g. Therefore the minimum value of f for $a > 0$ occurs for $a = 1/\sqrt{2}$. The minimum value of f is given by

$$f\left(\frac{1}{\sqrt{2}}\right) = \pi^4\, 2\left(\frac{3}{2}\right)^3 = \frac{27\pi^4}{4}$$

2. By hypothesis, if $t = 0$, then $x = 0 = y$. Thus $dx/dt = 0$ when $t = 0$, so that by the uniqueness theorem for differential equations, $x = 0$ for all t. Similarly, $dy/dt = 0$ when $t = 0$, so again by the uniqueness theorem, $y = 0$ for all t. Thus the entire orbit lies on the z axis. Finally, we notice that $dz/dt = -bz$, so that $z = c\,e^{-bt}$, where c is an arbitrary constant. Thus $\lim_{t\to\infty} z = \lim_{t\to\infty}(c\,e^{-bt}) = 0$. Consequently $\lim_{t\to\infty} X(t) = 0$.

3. Notice that when $x = 0$ we have $dx/dt = \sigma\, y$. Now if $y > 0$, then $dx/dt > 0$, so that x is increasing. By contrast, if $y < 0$, then $dx/dt < 0$, so that x is decreasing. For an orbit that revolves around the z axis, our observations imply that the orbit must revolve in a clockwise direction when viewed from above.

4. Notice that the inequality $-\lambda_2 > \lambda_1$ is equivalent to $0 > \lambda_1 + \lambda_2$, which by (3) and (4) is tantamount to the inequality $0 > -(\sigma+1)$, which is valid because $\sigma > 0$ by hypothesis. Next, the inequality $\lambda_1 > -\lambda_3$ is equivalent to

$$\frac{-(\sigma+1) + \sqrt{(\sigma-1)^2 + 4r\sigma}}{2} > b, \quad \text{that is,} \quad \sqrt{(\sigma-1)^2 + 4r\sigma} > 2b + \sigma + 1$$

Squaring both sides of the preceding inequality, and then simplifying, we find that

$$r\sigma - \sigma > b^2 + b + b\sigma$$

which is equivalent to the inequality $r > 1 + b(\sigma + 1 + b)/\sigma$.

5. If we let $x = -\sqrt{b(r-1)} + u$, $y = -\sqrt{b(r-1)} + v$, and $z = (r-1) + w$, then the Lorenz system is transformed into

$$\begin{aligned}
\frac{du}{dt} &= \sigma(-\sqrt{b(r-1)} + v) - \sigma(-\sqrt{b(r-1)} + u) = -\sigma u + \sigma v \\
\frac{dv}{dt} &= r(-\sqrt{b(r-1)} + u) - (-\sqrt{b(r-1)} + v) - (-\sqrt{b(r-1)} + u)(r - 1 + w) = u - v + \sqrt{b(r-1)}\, w - uw \\
\frac{dw}{dt} &= (-\sqrt{b(r-1)} + u)(-\sqrt{b(r-1)} + v) - b(r - 1 + w) = -\sqrt{b(r-1)}\, u - \sqrt{b(r-1)}\, v - bw + uv
\end{aligned}$$

By letting $F(u,v,w) = 0$, $G(u,v,w) = -uw$, and $H(u,v,w) = uv$, we see that the above system is almost linear at $(0,0,0)$. The associated matrix $A_{\mathbf{q}}$ is given by

$$A_{\mathbf{q}} = \begin{pmatrix} \sigma & \sigma & 0 \\ 1 & -1 & \sqrt{b(r-1)} \\ -\sqrt{b(r-1)} & -\sqrt{b(r-1)} & -b \end{pmatrix}$$

The characteristic equation of $A_{\mathbf{q}}$ turns out to be exactly the same as that for $A_{\mathbf{p}}$ given in (7). Thus both matrices have the same eigenvalues.

6. If we let $x = \sqrt{b(r-1)} + u$, $y = \sqrt{b(r-1)} + v$, and $z = (r-1) + w$, then as in the solution of Exercise 5, the Lorenz system is transformed into

$$\frac{du}{dt} = -\sigma u + \sigma v$$

$$\frac{dv}{dt} = u - v - \sqrt{b(r-1)}\, w - uw$$

$$\frac{dw}{dt} = \sqrt{b(r-1)}\, u + \sqrt{b(r-1)}\, v - bu + uv$$

By letting $F(u,v,w) = 0$, $G(u,v,w) = -uw$, and $H(u,v,w) = uv$, we see that the above system is almost linear at $(0,0,0$, and hence the Lorenz system is almost linear at $\mathbf{p}$. The above system yields the associated matrix appearing in (6).

7. The characteristic equation of $A_{\mathbf{p}}$ is

$$0 = \det \begin{pmatrix} \sigma - \lambda & \sigma & 0 \\ 1 & -1-\lambda & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b-\lambda \end{pmatrix}$$

which yields

$$0 = -(\sigma+\lambda)(1+\lambda)(b+\lambda) - \sigma b(r-1) + \sigma(b+\lambda) - (\sigma+\lambda)b(r-1) = -\lambda^3 - (\sigma+b+1)\lambda^2 - b(\sigma+r)\lambda - 2b\sigma(r-1)$$

which is equivalent to the formula in (7). By definition, the eigenvalues of $A_{\mathbf{p}}$ are the roots of the above equation.

8. Suppose that the roots of (7) are $c, \alpha + i\beta$, and $\alpha - i\beta$, where α and β are real numbers. Then

$$\lambda^3 + (\sigma+b+1)\lambda^2 + b(\sigma+r)\lambda + 2b\sigma(r-1) = (\lambda - c)(\lambda - \alpha - i\beta)(\lambda - \alpha + i\beta)$$

If we let $\lambda = 0$ in the equation, we obtain

$$2b\sigma(r-1) = -c(\alpha+i\beta)(\alpha-i\beta) = -c(a^2+b^2)$$

Since $b > 0$, $\sigma > 0$, and $r > 1$, it follows that $2b\sigma(r-1) > 0$. Therefore $c < 0$.

9. (a) If the characteristic equation of $A_{\mathbf{p}}$ has two (or three) real roots, then the polynomial on the left-hand side of (7) must have two distinct critical points. Setting the derivative to 0 yields

$$3\lambda^2 + 2(\sigma+b+1)\lambda + b(\sigma+r) = 0$$

There are two real solutions for this equation only if the discriminant is positive, that is, if

$$4(\sigma+b+1)^2 - 12b(\sigma+r) > 0$$

This inequality is equivalent to

$$r < \frac{\sigma^2 - b\sigma + b^2 + 2\sigma + 2b + 1}{3b}$$

(b) If $\sigma = 10$ and $b = 8/3$, then the final inequality in part (a) yields

$$r < \frac{100 - 80/3 + 64/9 + 20 + 16/3 + 1}{8} = \frac{961}{72}$$

Let $r_0 = 961/72$. Part (a) tells us that if $r > r_0$, then the characteristic polynomial cannot have two real roots. But that means that two of its roots must be complex numbers. Thus two eigenvalues of $A_{\mathbf{p}}$ are complex numbers.

10. The left-hand side of the equation factors as follows:

$$\lambda^3 + a\lambda^2 + c\lambda + ac = \lambda(\lambda^2+c) + a(\lambda^2+c) = (\lambda+a)(\lambda^2+c)$$

Thus the roots of the given equation are $-a$, $\sqrt{c}\,i$, and $-\sqrt{c}\,i$, which shows that two roots are pure imaginary.

11. In order to render (7) in the form of the equation in Exercise 10, we need to have

$$(\sigma+b+1)b(\sigma+r) = 2b\sigma(r-1)$$

Solving for r yields r_0 with the property that

$$r_0 = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$$

It follows from Exercise 10 that $A_{\mathbf{p}}$ has two pure imaginary eigenvalues if $r = r_0$.

12. The period-doubling occurs as r decreases from approximately 1.66 to 1.45.