Computational Fluid Mechanics and Heat Transfer

Solutions Manual

Chapter 2

2.1

The solution of Laplace's equation is

$$T(x,y)=\sum_{n=1}^{\infty} A_n \sin(n\pi x)\sinh[n\pi(y-1)]$$

To verify that the coefficient A_n given in Example 2.1 is correct, we can first use the boundary condition $T(x,0)=T_0$. Multiply this equation by $\sin(n\pi x)$, and integrate from 0 to 1:

$$\int_0^1 T_0 \sin(n\pi x)\,dx=\frac{T_0\left[1-(-1)^n\right]}{n\pi}=A_n \sinh(-n\pi)\frac{1}{2}$$

Using the trigonometry identity $\sinh(-x)=-\sinh(x)$ the coefficient becomes

$$A_n=\frac{2T_0\left[(-1)^n-1\right]}{n\pi\sinh(n\pi)}$$

2.2

For this problem, $F(r,\theta)=r-r_b=0$, thus $\nabla F=\mathbf{i}_r$ and the boundary condition is $u_r=\mathbf{V}\bullet\mathbf{i}_r=\nabla\phi\bullet\mathbf{i}_r=\dfrac{\partial\phi}{\partial r}=0$. Since $\phi=V_\infty r\cos\theta+K\cos\theta/r$, we have $u_r=\cos\theta\left(V_\infty-\dfrac{K}{r^2}\right)$. The quantity in parenthesis must vanish on the cylinder $(r=r_b)$ so $K=r_b^2V_\infty$ and the required velocity boundary condition is satisfied.

2.3

Classical separation of variable provides the general term $X(x)T(t)$. Substituting into the wave equation $y_{tt} = a^2 y_{xx}$ yields the following set of differential equations:

$$X'' + \alpha^2 X = 0 \qquad T'' + \alpha^2 a^2 T = 0$$

The boundary and initial conditions are

$$X(0) = X(l) = 0 \quad T(0) = \sin\left(\frac{\pi x}{l}\right) \quad T'(t) = 0$$

This leads to a solution

$$y(x,t) = \sum A_n \sin\left(\frac{an\pi\, t}{l}\right)\cos\left(\frac{n\pi x}{l}\right)$$

In this case, only one term of the expansion is necessary to satisfy the specified initial displacement. Applying the boundary conditions eliminates all but the first term in the series.

2.5

Applying the transformation to Equation 2.18a for the hyperbolic case results in the equation

$$-\frac{b^2 - 4ac}{a}\phi_{\xi\eta} + (e - d\lambda_1)\phi_\xi + (e - d\lambda_2) + f\phi_\eta = \phi = g(\xi,\eta)$$

2.6

Let $\lambda_2 = \dfrac{b}{2a}$ and $\lambda_1 = c$. These selections provide transformed coordinates that are linearly independent. The coefficient of the $\phi_{\xi\eta}$ term is

$$a\lambda_1^2 - b\lambda_2 + c = -b^2 + 4ac = 0$$

and the cross derivative coefficient is

$$2a(\lambda_1\lambda_2) - b(\lambda_1 + \lambda_2) + 2c = -b^2 + 4ac = 0$$

and the correct form is obtained.

2.7

The divergence theorem is $\iint_D \nabla^2 u dA = \int_B \frac{\partial u}{\partial n} dl$. Since the original equation is Laplace's equation on the domain D, the integral must vanish and substituting $r=1$ on the boundary yields

$$\int_B f(\theta)\,d\theta = \int_B \frac{\partial u}{\partial n}(1)\,d\theta = 0$$

2.8

(a) For the equation $y^2 u_{xx} - x^2 u_{yy} = 0$ we have $a = y^2, \quad b = 0, \quad c = -x^2, \quad b^2 - 4ac = 4x^2 y^2$.
The discriminant is positive so the equation is always hyperbolic except when $x = 0$ and $y = 0$.
For this isolated case, the equation is parabolic.

(b) Let $\xi = x^2 + y^2$ and $\eta = x^2 - y^2$. The equation is transformed to

$$2\left(\xi^2 - \eta^2\right) u_{\xi\eta} - \eta u_\xi + \xi u_\eta = 0$$

2.9

(a) $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$ The discriminant is zero so the equation is parabolic.
(b) $\xi = y + kx, \quad \eta = y - x$ assuming the second characteristic is a constant $k \neq 1$.
(c) $2v_x - 4w_x + 2w_y + 3u = 0$

$w_x - v_y = 0$

Letting $\mathbf{Z} = (v, w)$

$$[A]\mathbf{Z}_x + [C]\mathbf{Z}_y = [F]$$

where $[A] = \begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix}$, $[C] = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ and $[F] = \begin{bmatrix} -3u \\ 0 \end{bmatrix}$.

(d) $D = (4)^2 - 4(2)(2) = 0$ Therefore, the system of equations is parabolic.

2.10

Classify the system of equations:

$$\frac{\partial u}{\partial t}+8\frac{\partial v}{\partial x}=0$$

$$\frac{\partial v}{\partial t}+2\frac{\partial u}{\partial x}=0$$

$$a_1=1,\quad b_1=0,\quad c_1=0,\quad d_1=8$$

$$a_2=0,\quad b_2=1,\quad c_2=2,\quad d_2=0$$

Since $D>0$, the system of equations is hyperbolic.

2.11

Answer: The equation is elliptic for all values of $a\neq 0$.

2.12

Answer: hyperbolic

2.13

Answer: elliptic, hyperbolic

2.18

Answer: elliptic, hyperbolic

2.19

(a) $f(x) = \sin x, \;\; 0 \le x \le \pi$

Cosine series is $f(x) = \dfrac{a_0}{2} + \sum A_n \cos(n\pi x)$ where $n = 1 \to \infty$. In this series, all basis functions $\left[\cos(n\pi x)\right]$ are orthogonal to the function that is to be expanded. Thus, all of the Fourier coefficients vanish except $a_0 = \dfrac{2}{\pi}$. For the prescribed function, this is the best that can be done with the cosine series.

(b) In this case $f(x) = \cos x$ is itself the cosine series and only one term of the Fourier series survives.

2.20

(a) $a = 1, \quad b = 3, \quad c = 2$

$$\frac{dy}{dx} = 1, 2$$

(b) $a = 1, \quad b = 12, \quad c = 2$

$$\frac{dy}{dx} = -1$$

This is a parabolic equation and the other characteristic may be chosen with the restriction that the two are linearly independent.

2.21

(a) Hyperbolic $\lambda_1 = 2, \quad \lambda_2 = 1$. Let $\xi = y - x, \quad \eta = y + 2x$. This transforms to $u_{\xi\eta} = 0$.

(b) Parabolic $\lambda_1 = -1$. Let $\lambda_2 = 1, \quad \xi = y - x, \quad \eta = y + x$. This transforms to $u_{\xi\xi} = 0$.

2.22

(a) Answer: $u_{\xi\xi} + u_{\eta\eta} = 0$

(b) Answer: $u_{\xi\xi} + u_{\eta\eta} + \dfrac{u_\xi}{4} = 0$

2.23

(a) $\lambda_1 = -3$. Let $\lambda_2 = 1$, $\xi = y - x, \quad \eta = y + 3x$. After transforming we have $16u_{\xi\xi} - u_\xi + 3u_\eta - e^{xy} = 0$ where $x = \dfrac{\eta - \xi}{4}, y = \dfrac{\eta + 3\xi}{4}$.

2.24

Answer: $u(x, y) = -\sinh(y - \pi)\dfrac{\sin x}{\sinh \pi} - 2\sinh(y - \pi)\dfrac{\sin 2x}{\sinh 2\pi}$

2.25

Answer: $u(x, y) = -\dfrac{\sin x}{\sinh \pi} - \dfrac{2\sin 2x}{\sinh 2\pi}$

2.26

Answer: $u(x,y)=\sum_{n=1}^{\infty} A_n \sinh\left[n(y-\pi)\right]\sin(nx)$

where $A_n = -\dfrac{2}{n\sinh(n\pi)}\left[2\left(\pi^2-\dfrac{5}{n^2}-\dfrac{12}{n^2 n^4}\right)+\dfrac{2\cos(n\pi)}{n}\left(\dfrac{10\pi}{n^2}-\dfrac{24}{\pi n^4}\right)\right]$

2.27

Answer: $T(x,y)=e^{-4\pi^2 t}\sin(2\pi x)$
