

Chapter 2

Euclidean Space

2.1 Vectors

1. Determine $3\mathbf{u} + \mathbf{v} - 2\mathbf{w}$, where

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \text{ and } \mathbf{w} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

$$\text{Ans: } \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

2. Express the given vector equation as a system of linear equations.

$$x_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{Ans: } 3x_1 + 2x_2 &= 4 \\ -2x_1 + 5x_2 &= 1 \end{aligned}$$

3. Express the given vector equation as a system of linear equations.

$$x_1 \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{Ans: } 2x_1 - x_2 + 5x_3 &= -1 \\ 4x_2 + 3x_3 &= 2 \\ 5x_1 + 7x_2 &= 1 \end{aligned}$$

4. Express the given system of linear equations as a single vector equation.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 6 \\ -x_1 + x_3 &= 3 \end{aligned}$$

$$\text{Ans: } x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

5. Express the given system of linear equations as a single vector equation.

$$2x_1 + x_2 - 2x_3 = 1$$

$$-x_1 + x_2 + x_3 = 1$$

$$7x_1 + 3x_2 - x_3 = 1$$

$$\text{Ans: } x_1 \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

6. The general solution to a linear system is given. Express this solution as a linear combination of vectors.

$$x_1 = 3 - s_1$$

$$x_2 = s_1$$

$$\text{Ans: } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

7. The general solution to a linear system is given. Express this solution as a linear combination of vectors.

$$x_1 = 3 - s_1 + 3s_2$$

$$x_2 = s_1$$

$$x_3 = 3 + s_2$$

$$x_4 = s_2$$

$$\text{Ans: } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

8. Find the unknowns in the given vector equation.

$$2 \begin{bmatrix} 1 \\ a \end{bmatrix} + 4 \begin{bmatrix} b \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

$$\text{Ans: } a = 1, b = -1$$

9. Find the unknowns in the given vector equation.

$$2 \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} - \begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ c \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix}$$

$$\text{Ans: } a = 2, b = -1, c = 0$$

10. Express \mathbf{b} as a linear combination of the other vectors, if possible.

$$\mathbf{a}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 14 \\ -2 \end{bmatrix}$$

Ans: $3\mathbf{a}_1 - 2\mathbf{a}_2 = \mathbf{b}$

11. Express \mathbf{b} as a linear combination of the other vectors, if possible.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

Ans: $-2\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3 = \mathbf{b}$

True or False: If \mathbf{u} and \mathbf{v} are vectors, and c and d are scalars, then $c(d\mathbf{u} + \mathbf{v}) = (cd)\mathbf{u} + \mathbf{v}$.

Ans: False

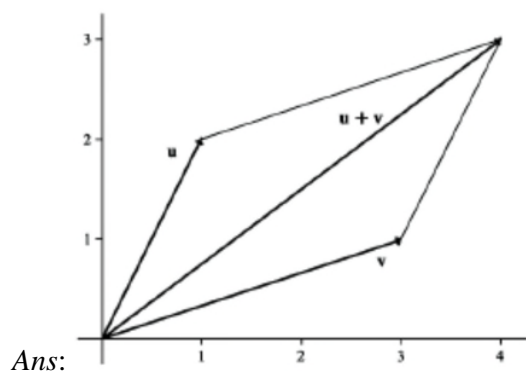
13. **True or False:** If \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors, then $\mathbf{u} - (\mathbf{v} + \mathbf{w}) = (\mathbf{u} - \mathbf{v}) + (\mathbf{u} - \mathbf{w})$.

Ans: False

14. **True or False:** If $\mathbf{u} + \mathbf{v} = \mathbf{w}$, then $\mathbf{v} = \mathbf{w} - \mathbf{u}$.

Ans: True

15. Sketch the graph of $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and then use the Parallelogram Rule to sketch the graph of $\mathbf{u} + \mathbf{v}$.



16. Determine how to divide a total mass of 18 kg among the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix} \text{ so that the center of mass is } \begin{bmatrix} 2/9 \\ 2/9 \\ 2/9 \end{bmatrix}.$$

Ans: Place 10 kg at \mathbf{u}_1 , 1 kg at \mathbf{u}_2 , and 7 kg at \mathbf{u}_3 .

17. Find an example of a linear system with two equations and three variables that has

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ as the general solution.}$$

Ans: A possible answer is

$$x_1 + x_2 - 5x_3 = 5$$

$$x_1 - x_2 - x_3 = -1$$

2.2 Span

1. Find four vectors that are in the span of the given vectors.

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Ans: For example, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

2. Find five vectors that are in the span of the given vectors.

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$$

Ans: For example, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 6 \\ 10 \\ 6 \end{bmatrix}$

3. Determine if \mathbf{b} is in the span of the other given vectors. If so, write \mathbf{b} as a linear combination of the other vectors.

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

Ans: $\mathbf{b} = \mathbf{a}_1 - \mathbf{a}_2$

4. Determine if \mathbf{b} is in the span of the other given vectors. If so, write \mathbf{b} as a linear combination of the other vectors.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

Ans: \mathbf{b} is not in the span of \mathbf{a}_1 and \mathbf{a}_2 .

5. Find A , \mathbf{x} , and \mathbf{b} such that $A\mathbf{x} = \mathbf{b}$ corresponds to the given linear system.

$$x_1 + x_2 - 2x_3 = 1$$

$$-x_1 + 2x_2 + 4x_3 = 8$$

Ans: $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$

6. Find A , \mathbf{x} , and \mathbf{b} such that $A\mathbf{x} = \mathbf{b}$ corresponds to the given linear system.

$$x_1 - x_3 = 1$$

$$2x_2 + 4x_3 = 3$$

$$-x_1 + 5x_3 = 1$$

$$\text{Ans: } A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ -1 & 0 & 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

7. Express the given system of linear equations as a vector equation.

$$2x_1 - x_3 + x_4 = 1$$

$$x_1 + 2x_2 + 4x_3 - x_4 = 0$$

$$\text{Ans: } x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

8. Determine if the columns of the given matrix span \mathbb{R}^2 .

$$\begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$$

Ans: Yes, the columns span \mathbb{R}^2 .

9. Determine if the columns of the given matrix span \mathbb{R}^3 .

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Ans: No, the columns do not span \mathbb{R}^3 .

10. Determine if the system $A\mathbf{x} = \mathbf{b}$ (where \mathbf{x} and \mathbf{b} have the appropriate number of components) has a solution for all choices of \mathbf{b} .

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

Ans: Yes, a solution exists.

11. Find all values of h such that the vectors span \mathbb{R}^2 .

$$\mathbf{a}_1 = \begin{bmatrix} h \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 2 \\ h \end{bmatrix}$$

Ans: All real numbers, except $h \neq \pm 2$.

12. For what value(s) of h do the given vectors span \mathbb{R}^3 ?

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ h \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Ans: All real numbers, except $h \neq 5$.

13. **True** or **False**: Suppose a matrix A has n rows and m columns, with $m < n$. Then the

columns of A do not span \mathbb{R}^n .

Ans: True

14. **True** or **False:** Suppose a matrix A has n rows and m columns, with $m > n$. Then the columns of A span \mathbb{R}^n .

Ans: False

15. **True** or **False:** If the columns of a matrix A with n rows and m columns do not span \mathbb{R}^n , then there exists a vector \mathbf{b} in \mathbb{R}^n such that $A\mathbf{x} = \mathbf{b}$ does not have a solution.

Ans: True

16. **True** or **False:** If the columns of a matrix A with n rows and m columns spans \mathbb{R}^n , then $m \geq n$.

Ans: True

2.3 Linear Independence

1. Determine if the given vectors are linearly independent.

$$\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Ans: Linearly independent

2. Determine if the given vectors are linearly independent.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ -4 \\ -8 \end{bmatrix}$$

Ans: Not linearly independent

3. Determine if the given vectors are linearly independent.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \end{bmatrix}$$

Ans: Linearly independent

4. Determine if the columns of the given matrix are linearly independent.

$$\begin{bmatrix} 3 & 2 \\ -2 & -3 \end{bmatrix}$$

Ans: Linearly independent

5. Determine if the columns of the given matrix are linearly independent.

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 0 \\ 3 & 2 & 2 \end{bmatrix}$$

Ans: Linearly independent

6. Determine if the columns of the given matrix are linearly independent.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Ans: Not linearly independent

7. Determine if the homogeneous system $Ax = 0$ has any nontrivial solutions, where

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 2 \\ 0 & 1 \end{bmatrix}.$$

Ans: $Ax = 0$ has only the trivial solution.

8. Determine if the homogeneous system $Ax = 0$ has any nontrivial solutions, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Ans: $Ax = 0$ has nontrivial solutions.

9. Determine by inspection (that is, with only minimal calculations) if the given vectors form a linearly dependent or linearly independent set. Justify your answer.

$$\mathbf{u} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 20 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Ans: Linearly dependent, by Theorem 2.14

10. Determine if one of the given vectors is in the span of the other vectors.

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

Ans: Yes, since $\mathbf{w} = -\mathbf{u} + \mathbf{v}$.

11. **True or False:** Suppose matrix A has n rows and m columns, with $n < m$. Then the columns of A are linearly dependent.

Ans: True

12. **True or False:** Suppose a matrix A has n rows and m columns, with $n \geq m$. Then the columns of A are linearly independent.

Ans: False

13. **True** or **False**: Suppose there exists a vector \mathbf{x} such that $A\mathbf{x} = \mathbf{b}$. Then the columns of A are linearly independent.

Ans: False

14. **True** or **False**: If $A\mathbf{x} \neq \mathbf{0}$ for every $\mathbf{x} \neq \mathbf{0}$, then the columns of A are linearly independent.

Ans: True

15. **True** or **False**: If $\{u_1, u_2\}$, $\{u_1, u_3\}$, and $\{u_2, u_3\}$ are all linearly independent, then $\{u_1, u_2, u_3\}$ is linearly independent.

Ans: False