Chapter 2

2.1 **a.** Spring constant, k**:** The change in the force per unit length change of the spring.

b. Coefficient of subgrade reaction, k':

Spring constant divided by the foundation contact area, $k' = \dfrac{k}{A}$

c. Undamped natural circular frequency: $\omega_n = \sqrt{\dfrac{k}{m}}$ rad/s

where m = mass = $\dfrac{W}{g}$

d. Undamped natural frequency: $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$ (in Hz)

Note: Circular frequency defines the rate of oscillation in term of radians per unit time; 2π radians being equal to one complete cycle of rotation.

e. Period, T: The time required for the motion to begin repeating itself.

f. Resonance: Resonance occurs when $\dfrac{\omega_n}{\omega} = 1$

g. Critical damping coefficient: $c_c = 2\sqrt{km}$

where k = spring constant; m = mass = $\dfrac{W}{g}$

h. Damping ratio: $D = \dfrac{c}{c_c} = \dfrac{c}{2\sqrt{km}}$

where c = viscous damping coefficient; c_c = critical damping coefficient

i. Damped natural frequency:

$$\omega_d = \omega_n\sqrt{1-D^2}$$

$$f_d = \sqrt{1-D^2}\, f_n$$

2.2 Weight of machine + foundation, $W = 400$ kN
Spring constant, $k = 100{,}000$ kN/m

Mass of the machine + foundation, $m = \dfrac{W}{g} = \dfrac{400}{9.81} = 40.77\ \dfrac{\text{kN}}{\text{m/s}^2}$

Natural frequency of undamped free vibration is [Eq. (2.19)]

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{100{,}000}{40.77}} = \mathbf{7.88\ Hz}$$

From Eq. (2.18), $T = \dfrac{1}{f_n} = \dfrac{1}{7.88} = \mathbf{0.127\ s}$

2.3 Weight of machine + foundation, $W = 400$ kN
Spring constant, $k = 100{,}000$ kN/m

Static deflection of foundation is [Eq. (2.2)]

$$z_s = \frac{W}{k} = \frac{400}{100{,}000} = 4\times10^{-3}\ \text{m} = \mathbf{4\ mm}$$

2.4 External force to which the foundation is subjected, $Q = 35.6\sin\omega t$ kN
$f = 13.33$ Hz
Weight of the machine + foundation, $W = 178$ kN
Spring constant, $k = 70{,}000$ kN/m

For this foundation, let time $t = 0$, $z = z_0 = 0$, $\dot{z} = v_0 = 0$

a. Mass of the machine + foundation, $m = \dfrac{W}{g} = \dfrac{178}{9.81} = 18.145\ \dfrac{\text{kN}}{\text{m/s}^2}$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{70{,}000}{18.145}} = 62.11\ \text{rad/s}$$

$$T = \frac{2\pi}{\omega_n} = \frac{2\pi}{62.11} = \mathbf{0.101\ s}$$

b. The frequency of loading, $f = 13.33$ Hz

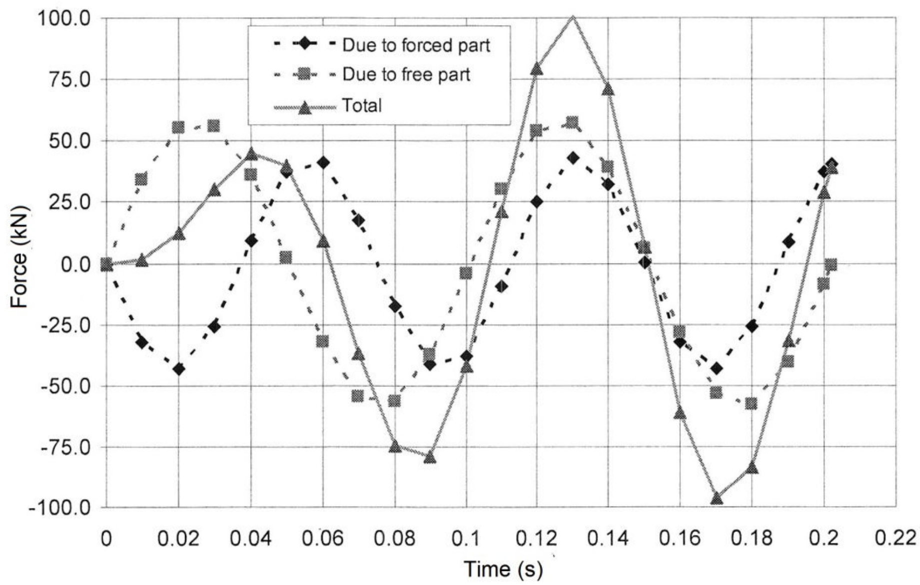
$$\omega = 2\pi f = 2\pi(13.33) = 83.75\ \text{rad/s}$$

Force due to forced part, $F_1 = k\left(\dfrac{Q_0/k}{1-\omega^2/\omega_n^2}\right)\sin\omega t$

$$= (70,000)\left(\frac{35.6/70,000}{1-83.75^2/62.11^2}\right)\sin(83.75t)$$

$$= \mathbf{43.51\sin(83.75\mathit{t})\ kN}$$

See the plot below for F_1 vs. t



c. Force due to free part, $F_2 = k\left(\dfrac{Q_0/k}{1-\omega^2/\omega_n^2}\right)\left(-\dfrac{\omega}{\omega_n}\sin\omega_n t\right)$

$$= 70,000\left(\frac{35.6/70,000}{1-83.75^2/62.11^2}\right)\left(-\frac{83.75}{62.11}\sin(62.11t)\right)$$

$$= \mathbf{58.67\sin(62.11\mathit{t})\ \ kN}$$

See the plot above in Part **b** for F_2 vs. t.

d. Total dynamic force on the subgrade:

$$F = F_1 + F_2 = \mathbf{-43.51\sin(83.75\mathit{t}) + 58.67(62.11\mathit{t})\ \ kN}$$

The plot of variation of the dynamic force on the subgrade of the foundation due to (a) forced part, (b) free part, and (c) total of the response for time $t = 0$ to $t = 2T$ is shown in the figure above (Part **b**).

2.5 The natural frequency of the undamped free vibration of the spring mass system is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}}$$ where k_{eq} = equivalent stiffness of the spring system

For springs attached in series, the equivalent stiffness is given by

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}, \text{ or } \frac{1}{k_{eq}} = \frac{k_1 k_2}{k_1 + k_2}$$

The natural frequency of the given undamped free vibration spring mass system is

$$\boldsymbol{f_n = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{k_1 + k_2} \times \frac{1}{m}}}$$

2.6 The natural frequency of the undamped free vibration of the spring mass system is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}}$$ where k_{eq} = equivalent stiffness of the spring system

For springs attached in parallel, the equivalent stiffness is given by

$k_{eq} = k_1 + k_2$

The natural frequency of the given undamped free vibration spring mass system is

$$\boldsymbol{f_n = \frac{1}{2\pi}\sqrt{\frac{(k_1 + k_2)}{m}}}$$

2.7 The natural frequency of the undamped free vibration of the spring mass system is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}}$$ where k_{eq} = equivalent stiffness of the spring system

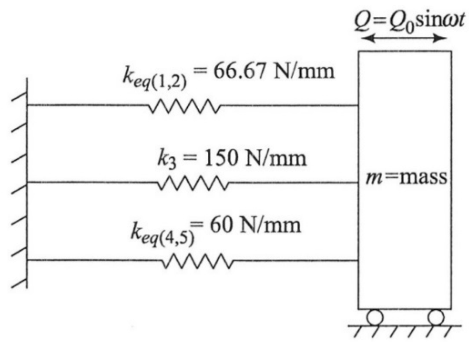
In the given spring-mass system, springs with stiffness k_1 and k_2 are in series.
Hence, their equivalent stiffness is

$$k_{eq(1,2)} = \frac{k_1 k_2}{k_1 + k_2} = \frac{100 \times 200}{100 + 200} = \frac{20{,}000}{300} = 66.67 \text{ N/mm}$$

Similarly, springs with stiffness k_4 and k_5 are in series. Hence, their equivalent stiffness is

$$k_{eq(4,5)} = \frac{k_4 k_5}{k_4 + k_5} = \frac{100 \times 150}{100 + 150} = 60 \text{ N/mm}$$

Now, the given spring system can be reduced to three springs in series.



The resulting system will be three springs in parallel. Their equivalent stiffness is given by

$$k_{eq} = k_{eq(1,2)} + k_3 + k_{eq(4,5)} = 66.67 + 150 + 60 = 276.67 \text{ N/mm} = 276.67 \text{ kN/m}$$

The natural frequency of the undamped free vibration of the spring mass system is given by

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}} = \frac{1}{2\pi}\sqrt{\frac{267.67 \times 1000}{100}} = \mathbf{8.37\ Hz}$$

Time period $T = (1/f_n) = (1/8.37) = \mathbf{0.119\,s}$

2.8 Sinusoidal–varying force, $Q = 50 \sin \omega t$ N; $Q_0 = 50$ N; $\omega = 47$ rad/s

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{276.67 \times 1000}{100}} = 52.6 \text{ rad/s}$$

Amplitude of vibration = static deflection z_s × magnification M

$$z_s = \frac{Q_0}{k_{eq}} = \frac{50}{276.67} = 0.1807 \text{ mm}$$

From Eq. (2.34),

$$M = \frac{1}{1 - (\omega/\omega_n)^2} = \frac{1}{1 - (47/52.6)^2} = 4.96$$

Amplitude of vibration = 0.1807 × 4.96 = **0.896 mm**

2.9 Weight of the body, $W = 135$ N
Mass of the body, $m = W/g = 135/9.81 = 13.76$ kg
Spring constant, $k = 2600$ N/m
Dashpot resistance, $c = 0.7/(60/1000) = 11.67$ N-s/m

a. Damped natural frequency [Eq. (2.67)]

$$f_d = \sqrt{1 - D^2}\, f_n$$

$$D = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{11.67}{2\sqrt{2600 \times 13.76}} = 0.031$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2600}{13.76}} = 2.19 \text{ Hz}$$

$$f_d = \sqrt{1 - 0.031^2} \times (2.19) = \mathbf{2.18\ Hz}$$

b. Damping ratio [Eq. (2.47b)],

$$D = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{11.67}{2\sqrt{2600 \times 13.76}} = \mathbf{0.031}$$

c. Ratio of successive amplitudes of the body is given by [Eq. (2.70)],

$$\frac{Z_n}{Z_{n+1}} = e^{\delta}$$

where $\delta = \ln\left(\frac{Z_n}{Z_{n+1}}\right) = \frac{2\pi D}{\sqrt{1 - D^2}} = \frac{2\pi \times 0.031}{\sqrt{1 - 0.031^2}} = 0.195$

$$\frac{Z_n}{Z_{n+1}} = e^{\delta} = e^{0.195} = \mathbf{1.215}$$

d. At time $t = 0$ s, amplitude $Z_0 = 25$ mm.
After n cycles of disturbance

$$\frac{1}{n}\ln\frac{Z_0}{Z_n} = \frac{2\pi D}{\sqrt{1 - D^2}}; \quad \ln\frac{Z_0}{Z_n} = \frac{2\pi n D}{\sqrt{1 - D^2}}$$

With $n = 5$,

$$\ln\frac{Z_0}{Z_5} = \frac{2\pi \times 5 \times D}{\sqrt{1 - D^2}} = \frac{2\pi \times 5 \times 0.031}{\sqrt{1 - 0.031^2}} = 0.974$$

$$\frac{Z_0}{Z_5} = e^{0.974} = 2.649; \quad Z_5 = \frac{25}{2.649} = 9.44 \text{ mm}$$

After 5 cycles of disturbance, the amplitude of vibration = **9.44 mm**

2.10 $Q_0 = 6.7$ kN
$\omega = 3100$ rad/min = 51.67 rad/s
Weight of machine + foundation, $W = 290$ kN
Spring constant, $k = 875$ MN/m = 875,000 kN/m

Natural angular frequency, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{875{,}000 \times 10^3}{290 \times 10^3/9.81}}$ = 172.04 rad/s

From Eq. (2.43), $F_{\text{dynam}} = \dfrac{Q_0}{1-(\omega/\omega_n)} = \dfrac{6.7}{1-(51.67/172.04)}$ = 9.58 kN

Maximum force on the subgrade = 290 + 9.58 = **299.58 kN**

Minimum force on the subgrade = 290 − 9.58 = **280.42 kN**

2.11 $Q_0 = 200$ kN
$\omega = 6000$ rad/min = 100 rad/s
Weight of machine + foundation, $W = 400$ kN
Spring constant, $k = 120{,}000$ kN/m

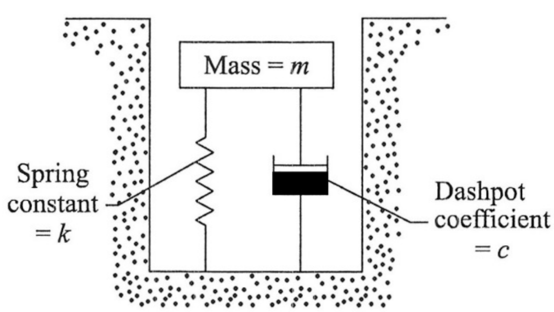
Natural angular frequency, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{120{,}000 \times 10^3}{400 \times 10^3/9.81}}$ = 54.25 rad/s

Dynamic force, $F_{\text{dynam}} = \dfrac{Q_0}{1-\omega/\omega_n} = \dfrac{200}{1-(100/54.25)}$ = 237.16 kN

Maximum force on the subgrade = 400 + 237.16 = **637.16 kN**

Minimum force on the subgrade = 400 − 237.16 = **162.84 kN**

2.12 Weight of the body, $W = 800$ kN
Spring constant, $k = 200{,}000$ kN/m
Dashpot coefficient, $c = 2340$ kN-s/m



a. $c_c = 2\sqrt{km} = 2\sqrt{200{,}000 \times 800/9.81}$ = **8077.1 kN-s/m**

b. Damping ratio, $D = \dfrac{c}{c_c} = \dfrac{2340}{8077.1}$ = **0.29**

c. $\delta = \dfrac{2\pi D}{\sqrt{1-D^2}} = \dfrac{2\pi \times 0.29}{\sqrt{1-0.29^2}}$ = **1.9**

d. $f_d = \sqrt{1-D^2}\, f_n$; $f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{200{,}000 \times 9.81}{800}}$ = 7.88 Hz

$f_d = \sqrt{1-0.29^2} \times 7.88$ = **7.54 Hz**

2.13 Weight of the body, W = 800 kN
Spring constant, k = 200,000 kN/m
Dashpot coefficient, c = 2340 kN-s/m
Q_0 = 25 kN
Operating frequency, ω = 100 rad/s

a. Natural circular frequency, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{200{,}000 \times 9.81}{800}}$ = 49.52 rad/s

From Problem 2.12, damping ratio, D = 0.29

From Eq. (2.28), the amplitude of vertical vibration of the foundation is

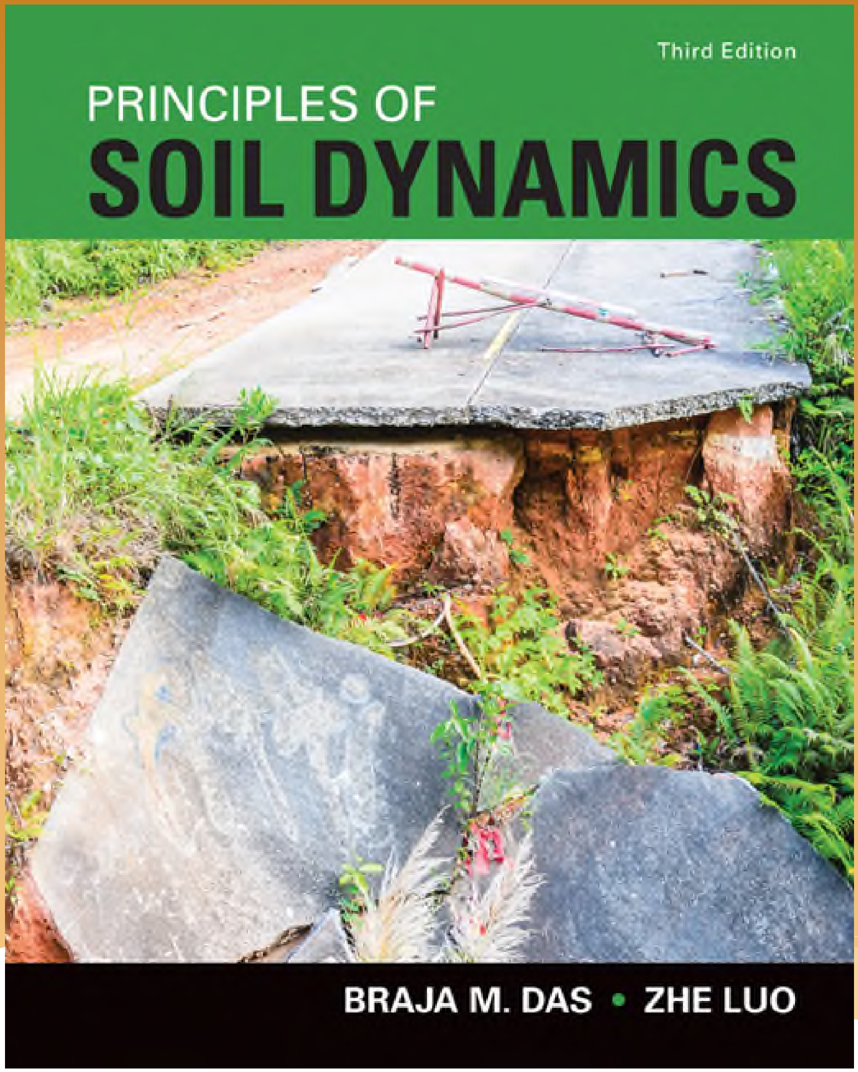
$$Z = \frac{(Q_0/k)}{\sqrt{[1-(\omega^2/\omega_n^2)]^2 + 4D^2(\omega^2/\omega_n^2)}}$$

$$= \frac{(25/200{,}000)}{\sqrt{[1-(100^2/49.52^2)]^2 + 4 \times 0.29^2(100^2/49.52^2)}}$$

$$= \mathbf{3.795 \times 10^{-5}\,m = 3.795 \times 10^{-2}\ mm}$$

b. From Eq. (2.90), the maximum dynamic force transmitted to the subgrade is

$$Z\sqrt{k^2 + (c\omega)^2} = (3.795 \times 10^{-5})\sqrt{200{,}000^2 + (2340 \times 100)^2} = \mathbf{11.68\ kN}$$



Chapter 2

Fundamentals of Vibration

2.1 Introduction

* Satisfactory design of foundations for vibrating equipment is mostly based on displacement considerations.
* Displacement due to vibratory loading can be classified under two major divisions:
  * Cyclic displacement due to the elastic response of the soil-foundation system to the vibrating loading
  * Permanent displacement due to compaction of soil below the foundation

2.1 Introduction Cont'd

* In order to estimate the cyclic displacement, it is essential to know the nature of the unbalanced forces.

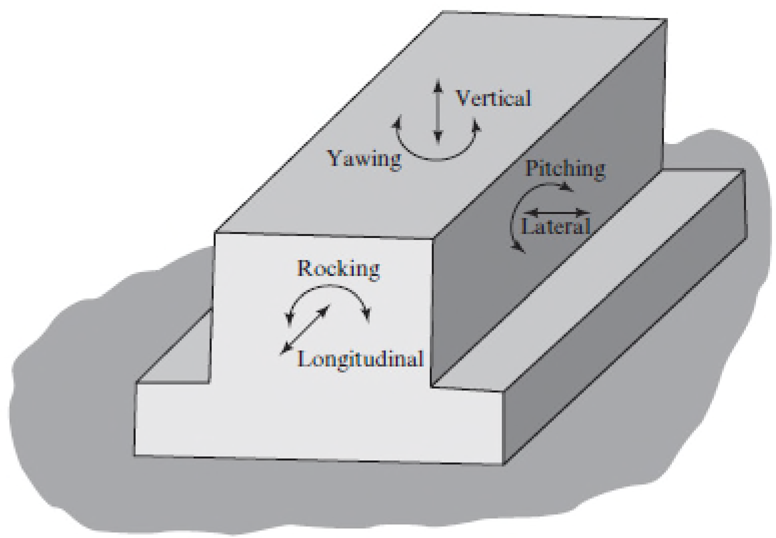


Figure 2.1 Six modes of vibration for foundation

2.1 Introduction Cont'd

- Note that a foundation can vibrate in any or all six possible modes.

- For ease of analysis, each mode is considered separately and design is carried out by considering the displacement due to each mode separately.

- Approximate mathematical models for computing the displacement of foundations under dynamic loads can be developed by treating soil as a viscoelastic material.

2.2 Fundamentals of Vibration

* Following are some essential definitions for the development of vibration theories:

  Free Vibration: Vibration of a system under the action of forces inherent in the system itself and in the absence of externally applied forces. The response of a system is called free vibration when it is disturbed and then left free to vibrate about some mean position.

  Forced Vibration: Vibration of a system caused by an external force. Vibrations that result from regular (rotating or pulsating machinery) and irregular (chemical process plant) exciting agencies are also called as forced vibrations.

2.2 Fundamentals of Vibration Cont'd

Degree of Freedom: The number of independent coordinates required to describe the solution of a vibrating system.

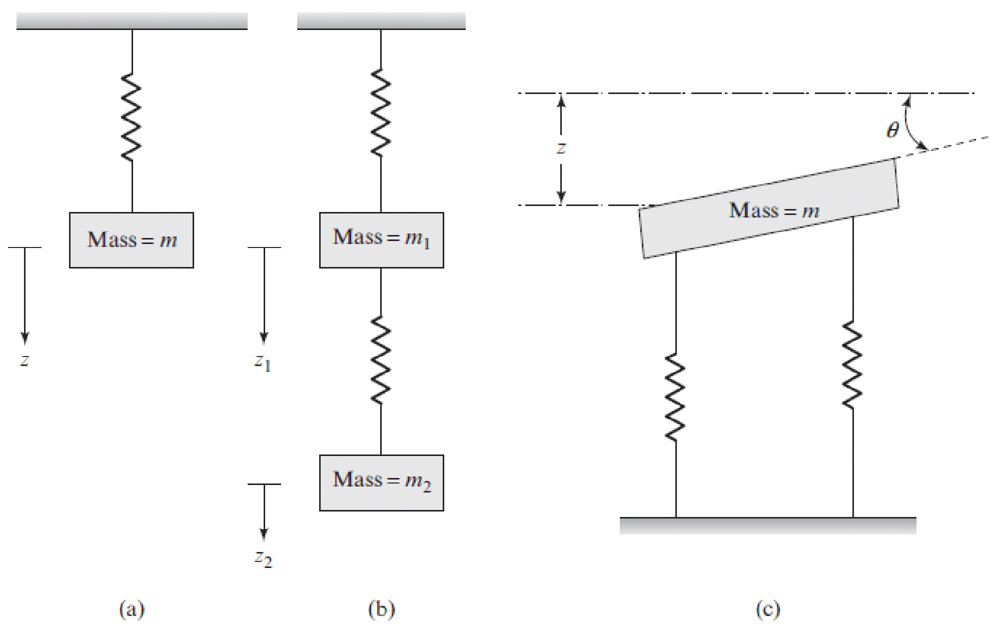


Figure 2.3 Degree of freedom for vibrating system

System with Single Degree of Freedom

2.3 Free Vibration of a Spring-Mass System

* Figure 2.4 shows a foundation resting on a spring. Let the spring represent the elastic properties of the soil. The load W represents the weight of the foundation plus that which comes from the machinery supported by the foundation.

* If the area of the foundation is equal to A, the intensity of load transmitted to the subgrade can be given by:

$$q = \frac{W}{A}$$

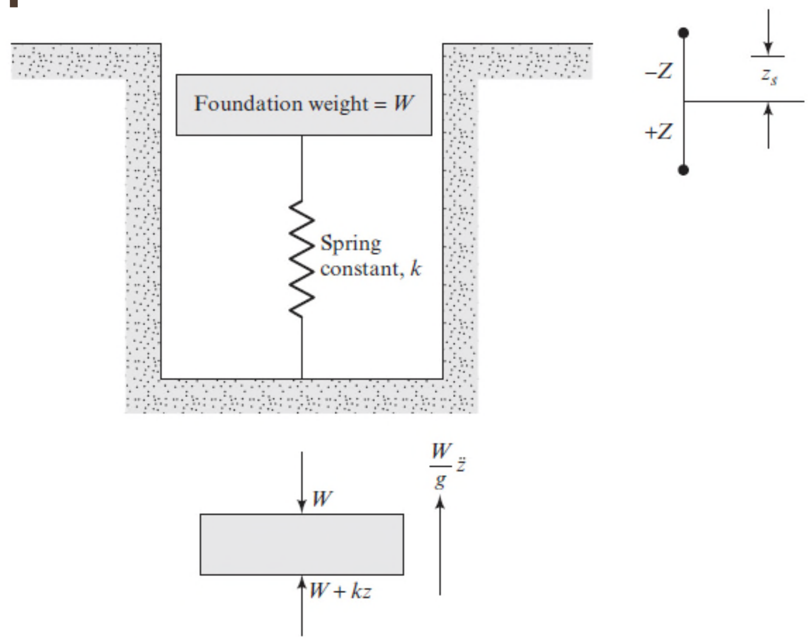


Figure 2.4 Free vibration of a mass-spring system

Due to the load W, a static deflection z_s will develop. By definition,

$$k = \frac{W}{z_s}$$ where k = spring constant for the elastic support.

2.3 Free Vibration of a Spring-Mass System Cont'd

- If the foundation is disturbed from its static equilibrium position, the system will vibrate.

- The equation of motion of the foundation when it has been disturbed through a distance z can be written from Newton's second law of motion as:

$$\ddot{z} + \left(\frac{k}{m}\right) z = 0$$

where

g = acceleration due to gravity
$\ddot{z} = d^2z/dt^2$
t = time
m = mass = W/g

2.3 Free Vibration of a Spring-Mass System Cont'd

* In order to solve the equation;

$$z = A_1 \cos \omega_n t + A_2 \sin \omega_n t$$

where A_1 and A_2 = constants

ω_n = undamped natural circular frequency

$$\omega_n = \sqrt{\frac{k}{m}}$$

The unit of ω_n is in radians per second (rad/s). Hence,

$$z = A_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + A_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

Displacement $z = z_0$

$$\text{Velocity} = \frac{dz}{dt} = \dot{z} = v_0$$

2.3 Free Vibration of a Spring-Mass System Cont'd

* Substitutions yield: $\boxed{z = Z\cos(\omega_n t - \alpha)}$ where $\boxed{\alpha = \tan^{-1}\left(\dfrac{v_0}{z_0\sqrt{k/m}}\right)}$

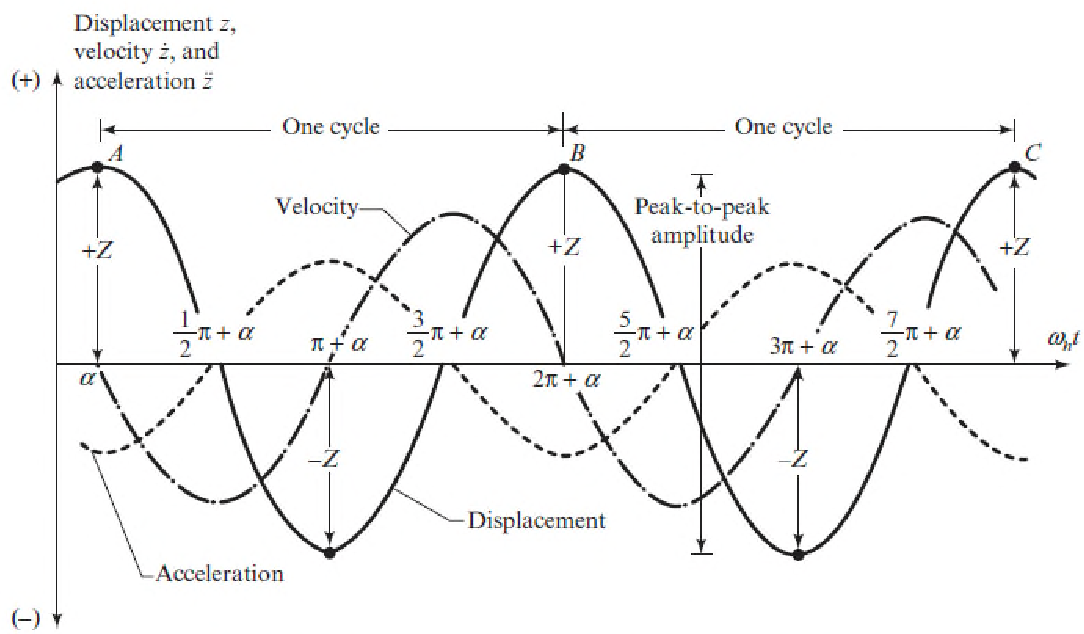


Figure 2.5 Plot of displacement, velocity, and acceleration for the free vibration of a mass-spring system
(Note: Velocity leads displacement by $\pi/2$ rad: acceleration leads velocity by $\pi/2$ rad.)

2.4 Forced Vibration of a Spring-Mass System

- Figure on the right shows a foundation that has been idealized to a simple spring-mass system. Weight *W* is equal to the weight of the foundation itself and that supported by it; the spring constant is *k*.
- This foundation is being subjected to an alternating force: $Q_0 \sin(\omega t + \beta)$
- This type of problem is generally encountered with foundations supporting reciprocating engines, and so on.
- The equation of motion for this problem can be given by: $m\ddot{z} + kz = Q_0 \sin(\omega t + \beta)$

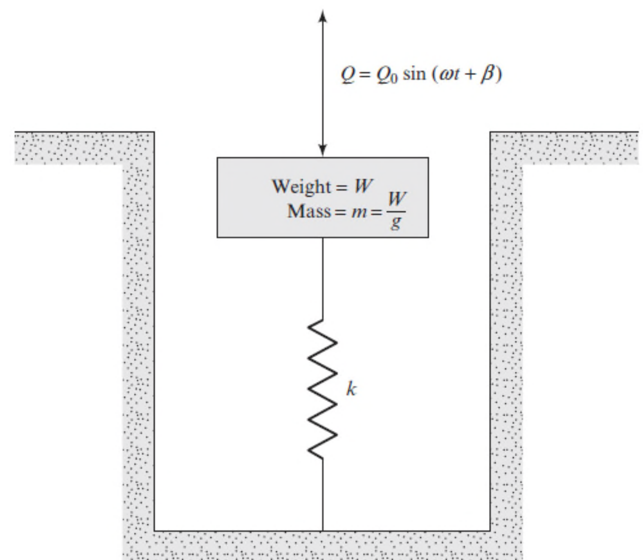
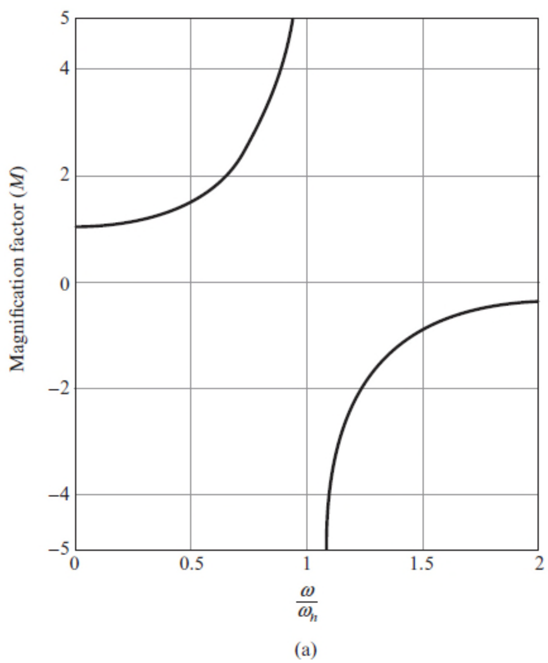


Figure 2.6 Forced vibration of mass-spring system title

2.4 Forced Vibration of a Spring-Mass System Cont'd

Maximum Force on Foundation Subgrade

The maximum and minimum force on the foundation subgrade will occur at the time when the amplitude is maximum, i.e., when velocity is equal to zero.



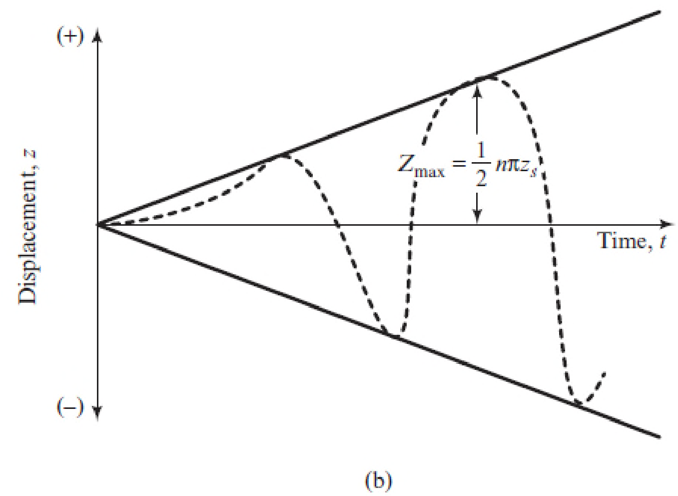


Figure 2.7 Forced vibration of a mass-spring system: (a) variation of magnification factor with ω/ω_n; (b) vibration of displacement with time at resonance ($\omega = \omega_n$)

2.4 Forced Vibration of a Spring-Mass System Cont'd

* For maximum deflection; $\dot{z} = 0$, or $\omega \cos \omega t - \omega \cos \omega_n t = 0$
* Since ω is not equal to zero,

$$\cos \omega t - \cos \omega_n t = 2 \sin\left(\frac{\omega_n - \omega}{2}\right) t \sin\left(\frac{\omega_n + \omega}{2}\right) t = 0$$

* In order to determine the maximum dynamic force, the maximum value of z_{max} is required:

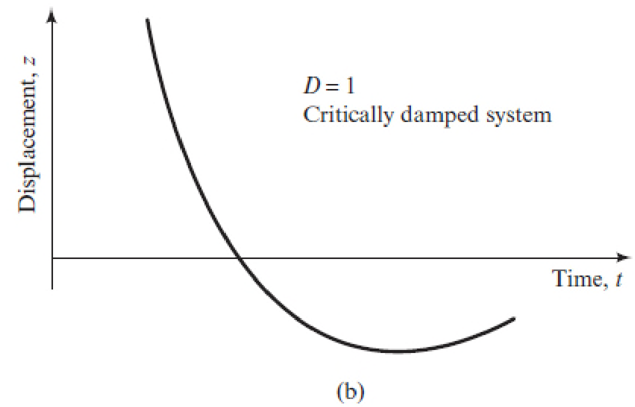
$$z_{\max(\max)} = \frac{(Q_0/k)}{1 - \omega/\omega_n}$$

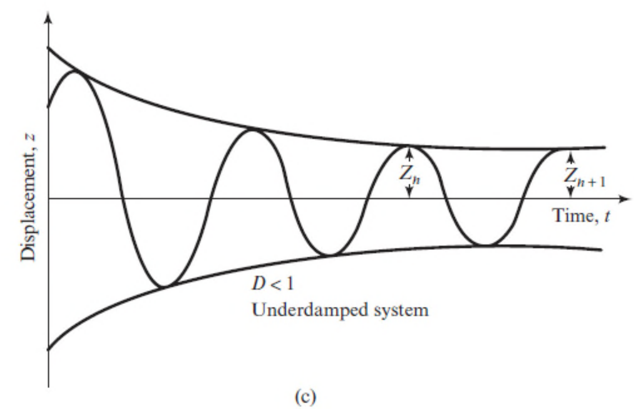
* Hence: $\boxed{F_{\text{dynam}(\max)} = k[z_{\max(\max)}] = \frac{k(Q_0/k)}{1-\omega/\omega_n} = \frac{Q_0}{1-\omega/\omega_n}}$
* The total force on the subgrade will vary between the limits

$$W - \frac{Q_0}{1-\omega/\omega_n} \text{ and } W + \frac{Q_0}{1-\omega/\omega_n}$$

2.5 Free Vibration with Viscous Damping

* In the case of undamped free vibration, vibration would continue once the system has been set in motion. However, in practical cases, all vibrations undergo a gradual decrease of amplitude with time. This characteristic of vibration is called *damping*.





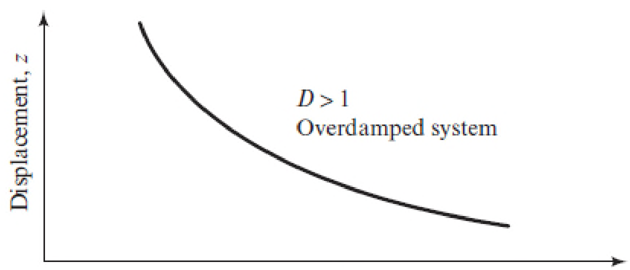


Figure 2.8 Free vibration of a mass-spring-dashpot system: (a) overdamped case; (b) critically damped case; (c) underdamped case

2.5 Free Vibration with Viscous Damping

* For free vibration of the foundation, the differential equation of motion can be given by:

$$m\ddot{z} + c\dot{z} + kz = 0 \qquad r^2 + \left(\frac{c}{m}\right)r + \frac{k}{m} = 0$$

$$r = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}$$

* There are three general conditions that may be developed from these equations:

1. If $c/2m > \sqrt{k/m}$, both roots of Eq. (2.45) are real and negative. This is referred to as an *overdamped* case.
2. If $c/2m = \sqrt{k/m}$, $r = -c/2m$. This is called the *critical damping* case. Thus, for this case,

$$\boxed{c = c_c = 2\sqrt{km}}$$

3. If $c/2m < \sqrt{k/m}$, the roots of Eq. (2.45) are complex:

$$r = -\frac{c}{2m} \pm i\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

This is referred to as a case of *underdamping*.

2.6 Steady-State Forced Vibration with Viscous Damping

* In the case of a foundation resting on a soil that can be approximated to an equivalent spring and dashpot, the foundation is being subjected to a sinusoidally varying force. The differential equation of motion for this system can be given as:

$$m\ddot{z} + kz + c\dot{z} = Q_0 \sin \omega t$$

* The transient part of the vibration is damped out quickly. Considering the particular solution for the steady-state motion, let

$$z = A_1 \sin \omega t + A_2 \cos \omega t$$

where A_1 and A_2 are constants.

* Therefore:

$$A_2 = \frac{-c\omega Q_0}{\left(k - m\omega^2\right)^2 + c^2\omega^2}$$ and; $$\boxed{z = Z\cos(\omega t + \alpha)}$$

2.6 Steady-State Forced Vibration with Viscous Damping Cont'd

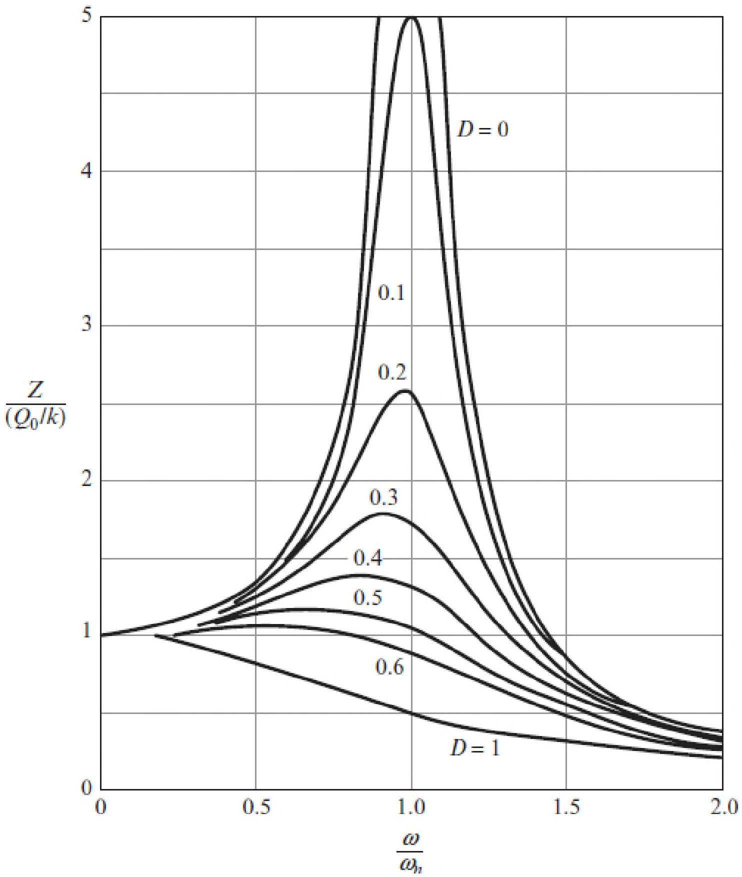


Figure 2.9 Plot of $Z/(Q_0/k)$ against ω/ω_n

* The amplitude of vibration at resonance can be obtained by:

$$Z_{res} = \frac{Q_0}{k}\frac{1}{\sqrt{\left[1-\left(1-2D^2\right)\right]^2+4D^2\left(1-2D^2\right)}}$$
$$= \frac{Q_0}{k}\frac{1}{2D\sqrt{1-D^2}}$$

2.6 Steady-State Forced Vibration with Viscous Damping Cont'd

Maximum Dynamic Force Transmitted to the Subgrade

For vibrating foundations, it is sometimes necessary to determine the dynamic force transmitted to the foundation. This can be given by summing the spring force and the damping force caused by relative motion between mass and dashpot; that is:

$$F_{\text{dynam}} = k\,z + c\,\dot{z}$$

If we let;

$$kZ = A\cos\phi \quad \text{and} \quad c\omega Z = A\sin\phi,$$

Then;

$$F_{\text{dynam}} = A\cos(\omega t + \phi + \alpha)$$

$$A = \sqrt{(A\cos\phi)^2 + (A\sin\phi)^2} = Z\sqrt{k^2 + (c\omega)^2}$$

Hence, the magnitude of maximum dynamic force will be: $Z\sqrt{k^2 + (c\omega)^2}$.

2.7 Rotating-Mass-Type Excitation

- In many cases of foundation equipment, vertical vibration of foundation is produced by counter-rotating masses. Since horizontal forces on the foundation at any instant cancel, the net vibrating force on the foundation can be determined to be equal to $2m_e e\omega^2 \sin\omega t$ (where m_e = mass of each counter-rotating element, e = eccentricity, and ω = angular frequency of the masses).

- In such cases, the equation of motion with viscous damping can be modified to the form:

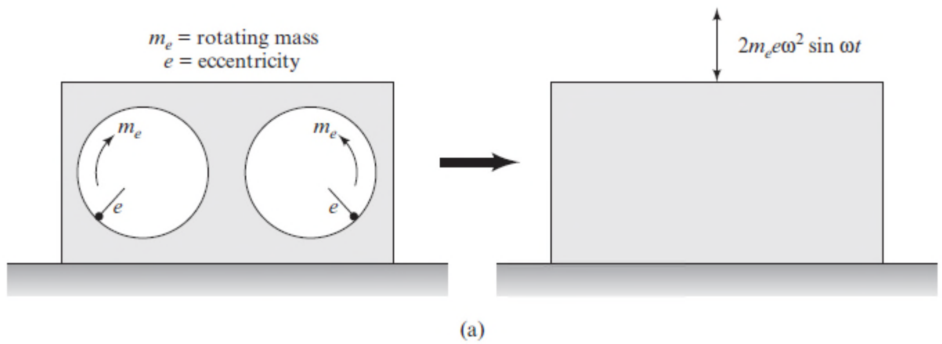
$$m\ddot{z} + kz + c\ddot{z} = Q_0 \sin\omega t$$

$$Q_0 = 2m_e e\omega^2 = U\omega^2$$

$$U = 2m_e e$$

- The solution for displacement is: $\boxed{z = Z\cos(\omega t + \alpha)}$

2.7 Rotating-Mass-Type Excitation Cont'd



* The angular resonant frequency for rotating-mass-type excitation can be obtained as:

$$f_m = \text{damped resonant frequency} = \frac{f_n}{\sqrt{1-2D^2}}$$

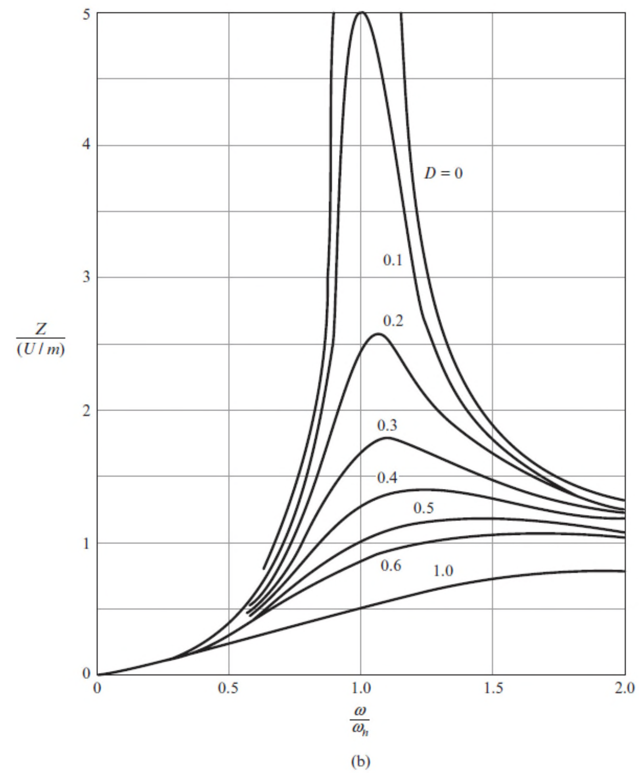


Figure 2.10 (a) Rotating mass-type excitation; (b) plot of $Z/(U/m)$ against ω/ω_n

2.8 Determination of Damping Ratio

* The damping ratio D can be determined from free and forced vibration tests on a system. In a free vibration test, the system is displaced from its equilibrium position, after which the amplitudes of displacement are recorded with time:

$$\delta = \ln\left(\frac{Z_n}{Z_{n+1}}\right) = \frac{2\pi D}{\sqrt{1-D^2}}$$

* If D is small, then

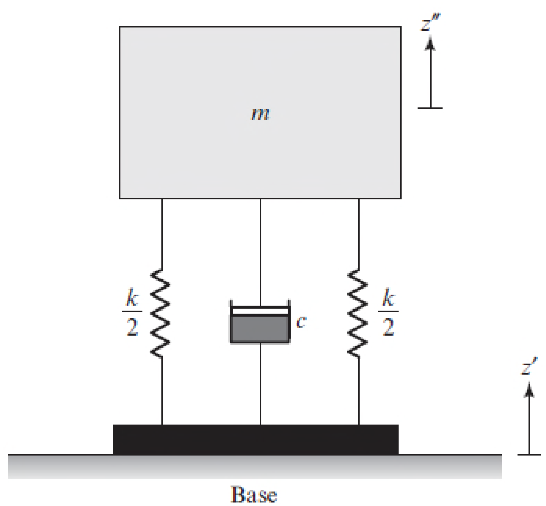
$$\delta = \ln\left(\frac{Z_n}{Z_{n+1}}\right) = 2\pi D$$

* Where Z_n = the peak amplitude of the nth cycle. Thus,

$$\boxed{D = \frac{1}{2\pi n}\ln\frac{Z_0}{Z_n}}$$

2.9 Vibration-Measuring Instrument

* Based on the theories of vibration presented in the preceding sections, it is now possible to study the principles of a vibration-measuring instrument, as shown below.



* The instrument consists of a spring-mass-dashpot system. It is mounted on a vibrating base. The relative motion of the mass *m* with respect to the vibrating base is monitored.

* Neglecting the transients let the absolute motion of the mass be given as:

$$z'' = Z'' \sin \omega t$$

2.9 Vibration-Measuring Instrument Cont'd

* So, the equation of motion for the mass can be written as:

$$m\ddot{z}'' + k(z'' - z') + c(\dot{z}'' - \dot{z}') = 0$$

$$\frac{Z}{Z'} = \frac{(\omega/\omega_n)^2}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + 4D^2(\omega/\omega_n)^2}}$$

* If the natural frequency of the instrument ω_n is small and ω/ω_n is large, then for practically all values of D, the magnitude of Z/Z' is about 1. Hence the instrument works as a *velocity pickup*.

* If D = 0.69 and ω/ω_n is varied from zero to 0.4, then:

$$\frac{Z}{\omega^2 Z'} \approx \frac{1}{\omega_n^2} = \text{const} \qquad \text{Thus;} \qquad Z \propto \omega^2 Z'$$

System with Two Degrees of Freedom

2.10 Vibration of a Mass-Spring System

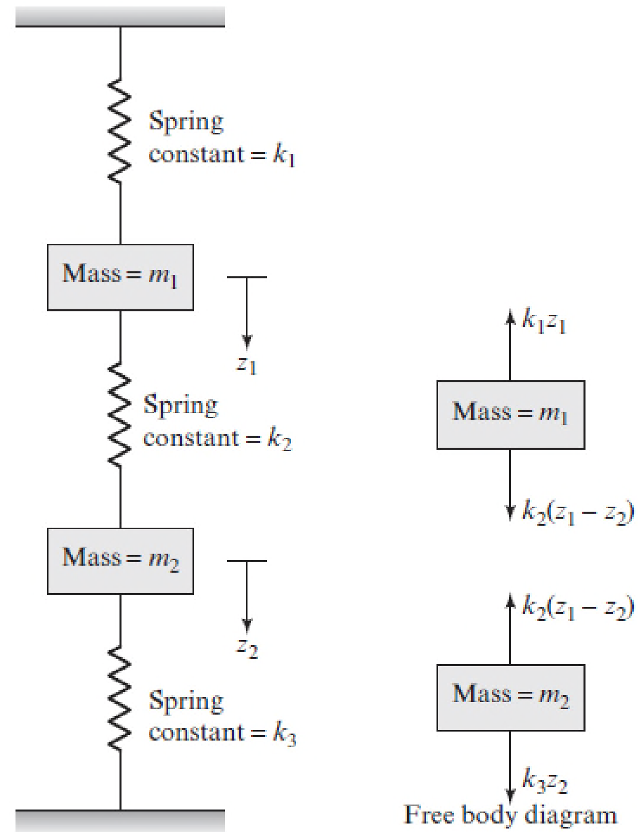
* A mass-spring with two degrees of freedom is shown on the right.

* If the masses m_1 and m_2 are displaced from their static equilibrium positions, the system will start to vibrate. The equations of motion of the two masses can be given as:

$$m_1\ddot{z}_1 + k_1 z_1 + k_2(z_1 - z_2) = 0$$

$$m_2\ddot{z}_2 + k_3 z_2 + k_2(z_2 - z_1) = 0$$

where m_1 and m_2 are the masses of the two bodies, k_1, k_2, and k_3 are the spring constants, and z_1 and z_2 are the displacements of masses m_1 and m_2.



2.10 Vibration of a Mass-Spring System Cont'd

* Let

$$z_1 = A\sin(\omega t + \alpha)$$

$$z_2 = B\sin(\omega t + \alpha)$$

* These equations yield to

$$(k_1 + k_2 - m_1\omega^2)A - k_2 B = 0$$

$$-k_2 A + (k_2 + k_3 - m_2\omega^2)B = 0$$

* The general equation of motion of the two masses can now be written as

$$z_1 = A_1 \sin\left(\omega_1 t + \alpha_1\right) + A_2 \sin\left(\omega_2 t + \alpha_2\right)$$

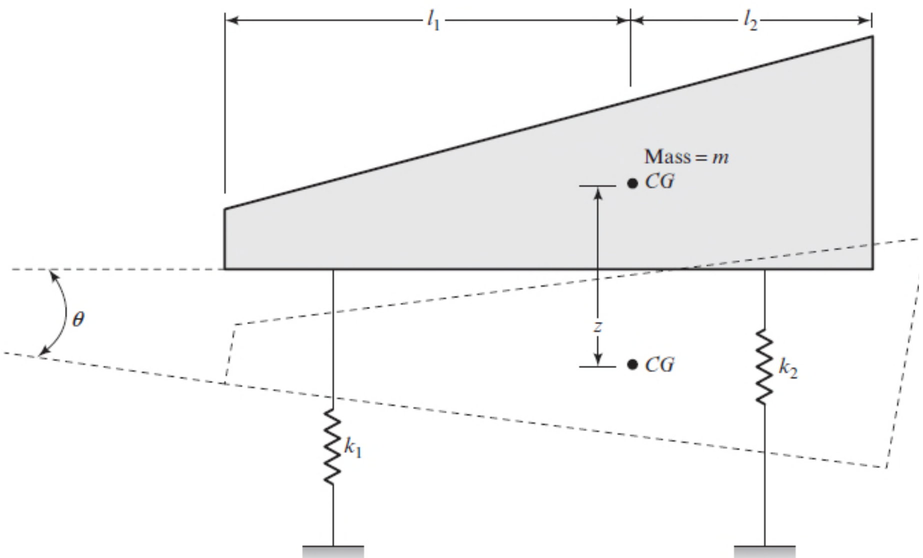
$$z_2 = B_1 \sin\left(\omega_1 t + \alpha_1\right) + B_2 \sin\left(\omega_2 t + \alpha_2\right)$$

2.11 Coupled Translation and Rotation of a Mass-Spring System (Free Vibration)

* The figure below shows a mass-spring system that will undergo translation and rotation. The equations of motion of the mass *m* can be given as

$$m\ddot{z} + k_1(z - l_1\theta) + k_2(z + l_2\theta) = 0$$

$$mr^2\ddot{\theta} - l_1k_1(z - l_1\theta) + l_2k_2(z + l_2\theta) = 0$$



where:

θ = angle of rotation of the mass m

$\ddot{\theta} = \dfrac{d^2\theta}{dt^2}$

r = radius of gyration of the body about the center of gravity (Note: $mr^2 = J$ = mass moment of inertia about the center of gravity)

k_1, k_2 = spring constants

z = distance of translation of the center of gravity of the body

2.11 Coupled Translation and Rotation of a Mass-Spring System (Free Vibration) Cont'd

* The general equations of motion can be given as:

$$z = Z_1 \cos \omega_{n_1} t + Z_2 \cos \omega_{n_2} t$$

and;

$$\theta = \Theta_1 \cos \omega_{n_1} t + \Theta_2 \cos \omega_{n_2} t$$

* The amplitude ratios can also be obtained:

$$\frac{Z_1}{\Theta_1} = -\frac{E_2}{E_1 - \cos \omega_{n_1}^2} = \frac{-(E_3 / r^2 - \omega_{n_1}^2)}{E_2 / r^2}$$

and;

$$\frac{Z_2}{\Theta_2} = -\frac{E_2}{E_1 - \cos \omega_{n_2}^2} = \frac{-(E_3 / r^2 - \omega_{n_2}^2)}{E_2 / r^2}$$