ELECTROMAGNETICS

PROBLEMS (in Chapters 1-14)

Solutions Manual

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P1 SOLUTIONS TO PROBLEMS
ELECTROSTATIC FIELD IN FREE SPACE

Section 1.1 Coulomb's Law

PROBLEM 1.1 **Three unequal charges in a triangle.** (a) The new situation is shown in Fig.P1.1(a), where charge 2 is now assumed to be $3Q$. From Coulomb's law, Eq.(1.1), magnitudes of the individual partial electric forces that charges 1, 2, and 3 exert on one another are

$$F_{e12} = F_{e21} = F_{e23} = F_{e32} = \frac{3Q^2}{4\pi\varepsilon_0 a^2} , \quad F_{e13} = F_{e31} = \frac{Q^2}{4\pi\varepsilon_0 a^2} , \tag{P1.1}$$

and all forces are repulsive. In the adopted xy-coordinate system, the resultant force on charge 1 is expressed as

$$\mathbf{F}_{e1} = \mathbf{F}_{e21} + \mathbf{F}_{e31} = F_{e21} \cos 60^\circ (-\hat{\mathbf{x}}) + F_{e21} \sin 60^\circ (-\hat{\mathbf{y}}) + F_{e31} (-\hat{\mathbf{x}})$$
$$= -\frac{Q^2}{8\pi\varepsilon_0 a^2} (5\,\hat{\mathbf{x}} + 3\sqrt{3}\,\hat{\mathbf{y}}) . \tag{P1.2}$$

Its magnitude, as well as the magnitude of the resultant force on charge 3, and the angle α in Fig.P1.1(a), determining the direction of both vectors $\mathbf{F}_{e1}$ and $\mathbf{F}_{e3}$, come out to be

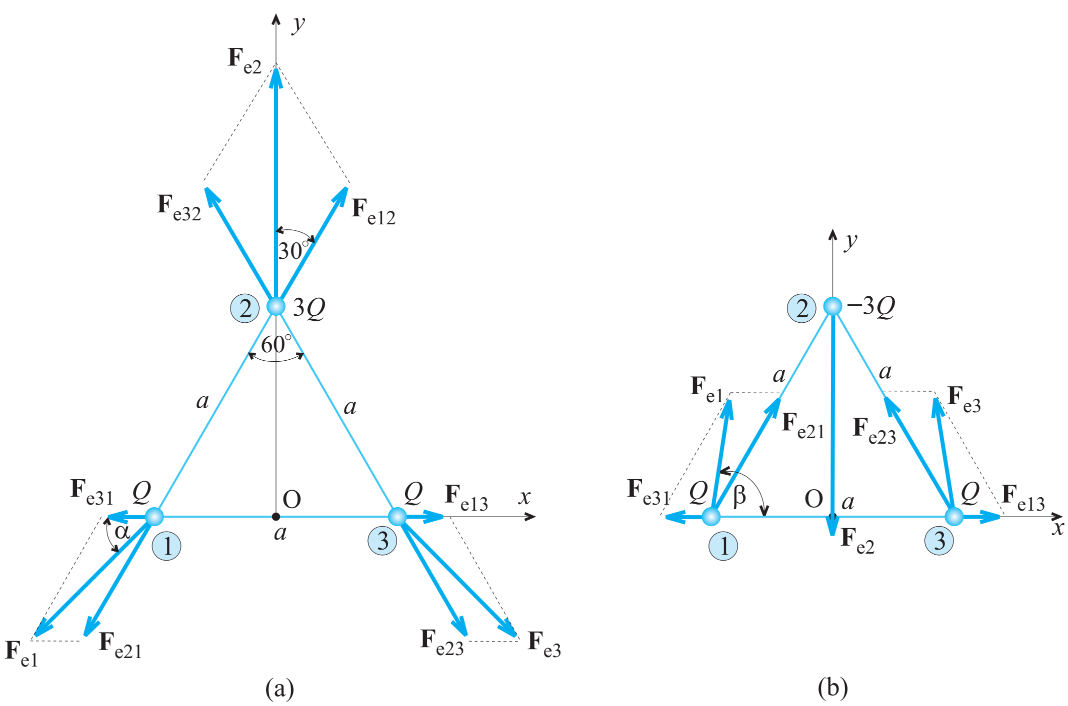

Figure P1.1 The same as in Fig.1.3(a) but with one of the point charges amounting to (a) $3Q$ and (b) $-3Q$.

$$F_{e1} = F_{e3} = \frac{Q^2}{8\pi\varepsilon_0 a^2}\sqrt{5^2 + (3\sqrt{3})^2} = \frac{\sqrt{13}\,Q^2}{4\pi\varepsilon_0 a^2}\,, \quad \alpha = \arctan\frac{3\sqrt{3}}{5} = 46.1^\circ \quad \text{(P1.3)}$$

($\arctan \equiv \tan^{-1}$).

The resultant electric force on charge 2 is, as in Fig.1.3(b) and Eq.(1.10), given by

$$F_{e2} = 2F_{e12}\cos 30^\circ = F_{e12}\sqrt{3} = \frac{3\sqrt{3}\,Q^2}{4\pi\varepsilon_0 a^2}\,, \quad \text{(P1.4)}$$

and $\mathbf{F}_{e2}$ is directed upward (perpendicularly to the opposite side of the equilateral triangle), that is, $\mathbf{F}_{e2} = F_{e2}\,\hat{\mathbf{y}}$, in Fig.P1.1(a).

(b) For charge 2 amounting to $-3Q$, Fig.P1.1(b), we have

$$\mathbf{F}_{e1} = F_{e21}\cos 60^\circ\,\hat{\mathbf{x}} + F_{e21}\sin 60^\circ\,\hat{\mathbf{y}} + F_{e31}(-\,\hat{\mathbf{x}}) = \frac{Q^2}{8\pi\varepsilon_0 a^2}\left(\hat{\mathbf{x}} + 3\sqrt{3}\,\hat{\mathbf{y}}\right)\,, \quad \text{(P1.5)}$$

and hence

$$F_{e1} = F_{e3} = \frac{Q^2}{8\pi\varepsilon_0 a^2}\sqrt{1 + (3\sqrt{3})^2} = \frac{\sqrt{7}\,Q^2}{4\pi\varepsilon_0 a^2}\,, \quad \beta = \arctan 3\sqrt{3} = 79.1^\circ\,. \quad \text{(P1.6)}$$

The force F_{e2} (magnitude) is the same as in Eq.(P1.4), and the direction of $\mathbf{F}_{e2}$ is downward, $\mathbf{F}_{e2} = F_{e2}(-\,\hat{\mathbf{y}})$.

PROBLEM 1.2 **Three charges in equilibrium.** The resultant Coulomb force on charge 3 in Fig.1.50 being zero, we have that

$$\mathbf{F}_{e3} = \mathbf{F}_{e13} + \mathbf{F}_{e23} = 0 \quad\longrightarrow\quad F_{e13} = F_{e23} \quad\longrightarrow\quad \frac{Q_1 Q_3}{4\pi\varepsilon_0 d^2} = \frac{Q_2 Q_3}{4\pi\varepsilon_0 (D-d)^2}$$

$$\longrightarrow\quad (D-d)^2 = \frac{Q_2}{Q_1}\,d^2 \quad\longrightarrow\quad D - d = \pm\sqrt{\frac{Q_2}{Q_1}}\,d \quad\longrightarrow\quad d = 2\ \text{cm}\,, \quad \text{(P1.7)}$$

where we eliminate the other solution, $d = 6$ cm, because $d < D$.

From the condition that the total force on charge 1 must also be zero,

$$F_{e21} + F_{e31} = 0 \quad\longrightarrow\quad \frac{Q_1 Q_2}{4\pi\varepsilon_0 D^2} = -\frac{Q_1 Q_3}{4\pi\varepsilon_0 d^2} \quad\longrightarrow\quad Q_3 = -\frac{d^2}{D^2}\,Q_2 = -4\ \text{pC}\,.$$

The condition $\mathbf{F}_{e2} = 0$ gives the same result.

PROBLEM 1.3 **Four charges at rectangle vertices.** With reference to Fig.P1.2,

$$\mathbf{F}_{e14} = F_{e14}\,\hat{\mathbf{x}}\,, \quad F_{e14} = \frac{Q^2}{4\pi\varepsilon_0 a^2}\,, \quad \text{(P1.8)}$$

$$\mathbf{F}_{e24} = F_{e24}\cos\alpha\,\hat{\mathbf{x}} + F_{e24}\sin\alpha\,\hat{\mathbf{y}}\,, \quad F_{e24} = \frac{Q^2}{4\pi\varepsilon_0 c^2}\,,$$

$$c = \sqrt{a^2 + b^2}\,, \quad \cos\alpha = \frac{a}{c}\,, \quad \sin\alpha = \frac{b}{c}\,, \quad \text{(P1.9)}$$

$$\mathbf{F}_{e34} = F_{e34}\,\hat{\mathbf{y}} \ , \quad F_{e34} = \frac{Q^2}{4\pi\varepsilon_0 b^2} \ , \tag{P1.10}$$

so that the total electric force on charge 4 amounts to

$$\mathbf{F}_{e4} = \mathbf{F}_{e14} + \mathbf{F}_{e24} + \mathbf{F}_{e34} = \frac{Q^2}{4\pi\varepsilon_0}\left[\left(\frac{1}{a^2} + \frac{a}{c^3}\right)\hat{\mathbf{x}} + \left(\frac{b}{c^3} + \frac{1}{b^2}\right)\hat{\mathbf{y}}\right]$$

$$= (9.637\,\hat{\mathbf{x}} + 24.48\,\hat{\mathbf{y}})\ \mu\text{N} \ . \tag{P1.11}$$

The magnitude of this vector comes out to be $|\mathbf{F}_{e4}| = 26.31\ \mu$N; $\mathbf{F}_{e4}$ makes an angle of $\beta = 68.5°$ with the x-axis (Fig.P1.2).

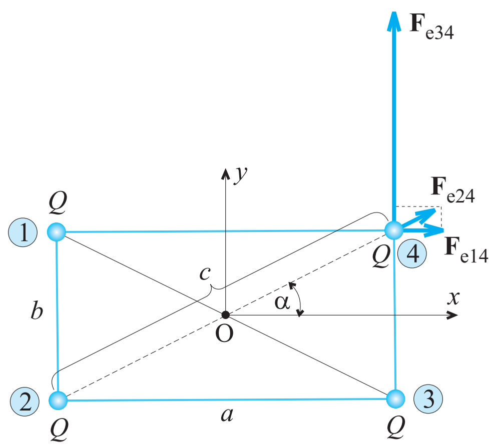

Figure P1.2 Computing the electric forces on charges placed at vertices of a rectangle.

Similarly, the resultant forces on other charges are

$$\mathbf{F}_{e1} = (-9.637\,\hat{\mathbf{x}} + 24.48\,\hat{\mathbf{y}})\ \mu\text{N} \ , \quad \mathbf{F}_{e2} = -(9.637\,\hat{\mathbf{x}} + 24.48\,\hat{\mathbf{y}})\ \mu\text{N} = -\mathbf{F}_{e4} \ ,$$

$$\mathbf{F}_{e3} = (9.637\,\hat{\mathbf{x}} - 24.48\,\hat{\mathbf{y}})\ \mu\text{N} = -\mathbf{F}_{e1} \ . \tag{P1.12}$$

These vectors are also shown in Fig.P1.2.

PROBLEM 1.4 **Five charges in equilibrium.** Because of symmetry, the resultant electric force on the charge (small ball) Q_2 at the square center is zero, as indicated in Fig.P1.3, so it is always in the electrostatic equilibrium, regardless of the actual value of Q_2 (which is unknown). Because of symmetry as well, if we find Q_2 such that one of the four charges (balls) at the square vertices is in the equilibrium, then this condition automatically applies to the remaining three balls with charge Q_1.

From Fig.P1.3, Q_2 that makes the resultant force on the lower left charge (charge 1) be zero obviously must be negative and is obtained as follows:

$$\mathbf{F}_{e1} = \mathbf{F}_{e21} + \mathbf{F}_{e31} + \mathbf{F}_{e41} + \mathbf{F}_{e51} = 0 \quad \longrightarrow \quad 2F_{e21}\cos 45° + F_{e31} = F_{e51}$$

$$\longrightarrow \quad 2\,\frac{Q_1^2}{4\pi\varepsilon_0 a^2}\,\frac{\sqrt{2}}{2} + \frac{Q_1^2}{4\pi\varepsilon_0(\sqrt{2}a)^2} = -\frac{Q_1 Q_2}{4\pi\varepsilon_0(\sqrt{2}a/2)^2} \qquad (Q_2 < 0)$$

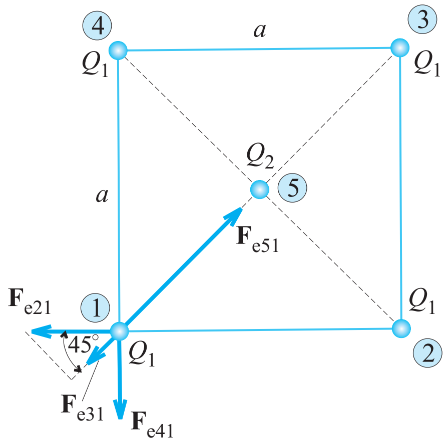

Figure P1.3 Finding the charge Q_2 at the center of a square such that all the five charges (small charged balls) are in the electrostatic equilibrium.

$$\longrightarrow \quad Q_2 = -\frac{(1+2\sqrt{2})Q_1}{4} = -4.785 \text{ pC} . \tag{P1.13}$$

PROBLEM 1.5 **Three point charges in space.** (a) With reference to Fig.P1.4(a), the partial electric force on the charge Q_2 due to the charge Q_1 is computed as

$$\mathbf{F}_{e12} = F_{e12}\sin 45^\circ\,\hat{\mathbf{x}} - F_{e12}\cos 45^\circ\,\hat{\mathbf{y}} = \frac{Q_1|Q_2|}{4\pi\varepsilon_0 R^2}\,\frac{\sqrt{2}}{2}\,(\hat{\mathbf{x}} - \hat{\mathbf{y}}) = 6.36(\hat{\mathbf{x}} - \hat{\mathbf{y}}) \text{ mN} . \tag{P1.14}$$

Of course, $\mathbf{F}_{e12} = -\mathbf{F}_{e21}$, with $\mathbf{F}_{e21}$ being given in Eq.(1.13) and Fig.1.5(b). Similarly, the force on Q_2 due to Q_3 comes out to be [Fig.P1.4(a)]

$$\mathbf{F}_{e32} = -F_{e32}\cos 45^\circ\,\hat{\mathbf{y}} + F_{e32}\sin 45^\circ\,\hat{\mathbf{z}} = \frac{Q_3|Q_2|}{4\pi\varepsilon_0 R^2}\,\frac{\sqrt{2}}{2}\,(-\hat{\mathbf{y}} + \hat{\mathbf{z}}) = 12.72(-\hat{\mathbf{y}} + \hat{\mathbf{z}}) \text{ mN} , \tag{P1.15}$$

so that the resultant force on this charge equals

$$\mathbf{F}_{e2} = \mathbf{F}_{e12} + \mathbf{F}_{e32} = 6.36(\hat{\mathbf{x}} - 3\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{z}}) \text{ mN} . \tag{P1.16}$$

Its magnitude is $F_{e2} = 6.36\sqrt{1+3^2+2^2}$ mN $= 23.8$ mN.

(b) The force on the charge Q_3 due to Q_1, Fig.P1.4(b), amounts to

$$\mathbf{F}_{e13} = -\mathbf{F}_{e31} = \frac{Q_1 Q_3}{4\pi\varepsilon_0 R^2}\,\frac{\sqrt{2}}{2}\,(-\hat{\mathbf{x}} + \hat{\mathbf{z}}) = 6.36(-\hat{\mathbf{x}} + \hat{\mathbf{z}}) \text{ mN} , \tag{P1.17}$$

and that due to Q_2 is

$$\mathbf{F}_{e23} = -\mathbf{F}_{e32} = 12.72(\hat{\mathbf{y}} - \hat{\mathbf{z}}) \text{ mN} . \tag{P1.18}$$

The resultant force on Q_3 is thus

$$\mathbf{F}_{e3} = \mathbf{F}_{e13} + \mathbf{F}_{e23} = 6.36(-\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} - \hat{\mathbf{z}}) \text{ mN} , \tag{P1.19}$$

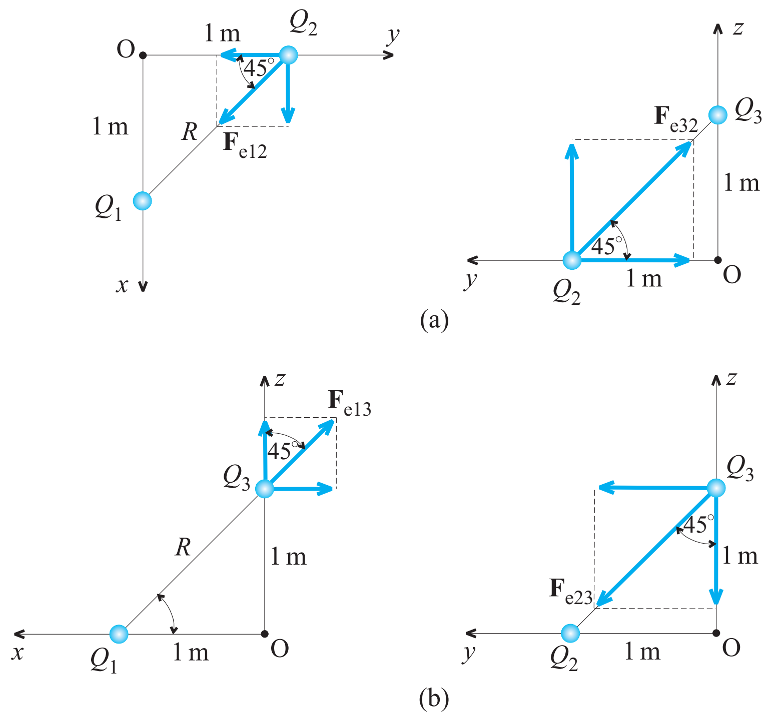

Figure P1.4 Calculating the resultant electric forces on the charge Q_2 (a) and Q_3 (b) in the system with three unequal point charges in space in Fig.1.5(a).

and its magnitude is $F_{e3} = 6.36\sqrt{1+2^2+1}$ mN $= 15.6$ mN.

(c) Combining Eqs.(1.16), (P1.16), and (P1.19), the sum of all the three resultant forces, on the three charges in Fig.1.5(a), is zero, namely,

$$\mathbf{F}_{e1} + \mathbf{F}_{e2} + \mathbf{F}_{e3} = 6.36(\hat{\mathbf{y}} - \hat{\mathbf{z}})\ \text{mN} + 6.36(\hat{\mathbf{x}} - 3\,\hat{\mathbf{y}} + 2\,\hat{\mathbf{z}})\ \text{mN}$$
$$+6.36(-\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} - \hat{\mathbf{z}})\ \text{mN} = 0\,, \tag{P1.20}$$

as expected. This is also obvious from

$$\mathbf{F}_{e1} + \mathbf{F}_{e2} + \mathbf{F}_{e3} = (\mathbf{F}_{e21} + \mathbf{F}_{e31}) + (\mathbf{F}_{e12} + \mathbf{F}_{e32}) + (\mathbf{F}_{e13} + \mathbf{F}_{e23})$$
$$= (\mathbf{F}_{e12} + \mathbf{F}_{e21}) + (\mathbf{F}_{e13} + \mathbf{F}_{e31}) + (\mathbf{F}_{e23} + \mathbf{F}_{e32}) = 0\,. \tag{P1.21}$$

PROBLEM 1.6 **Five charges at pyramid vertices.** See Example 1.4. Similarly to Eqs.(1.19)-(1.22), the resultant electric force on the top charge (charge 5) is given by

$$\mathbf{F}_{e5} = \mathbf{F}_{e15} + \mathbf{F}_{e25} + \mathbf{F}_{e35} + \mathbf{F}_{e45} = 4\left(|\mathbf{F}_{e15}|\cos\alpha\right)(-\hat{\mathbf{z}}) = -4\,\frac{Q^2}{4\pi\varepsilon_0 a^2}\,\cos 45^\circ\,\hat{\mathbf{z}}$$
$$= -\frac{\sqrt{2}Q^2}{2\pi\varepsilon_0 a^2}\,\hat{\mathbf{z}} \qquad \left(\sin\alpha = \frac{a\sqrt{2}/2}{a} = \frac{\sqrt{2}}{2} \quad\longrightarrow\quad \alpha = 45^\circ\right), \tag{P1.22}$$

where the adopted z-axis coincides with the axis of the pyramid containing this charge (vertical axis) and is directed upward.

PROBLEM 1.7 **Eight charges at cube vertices.** Referring to Fig.P1.5(a), the resultant force on charge 8 of the cube due to charges 1, 2, and 3 equals

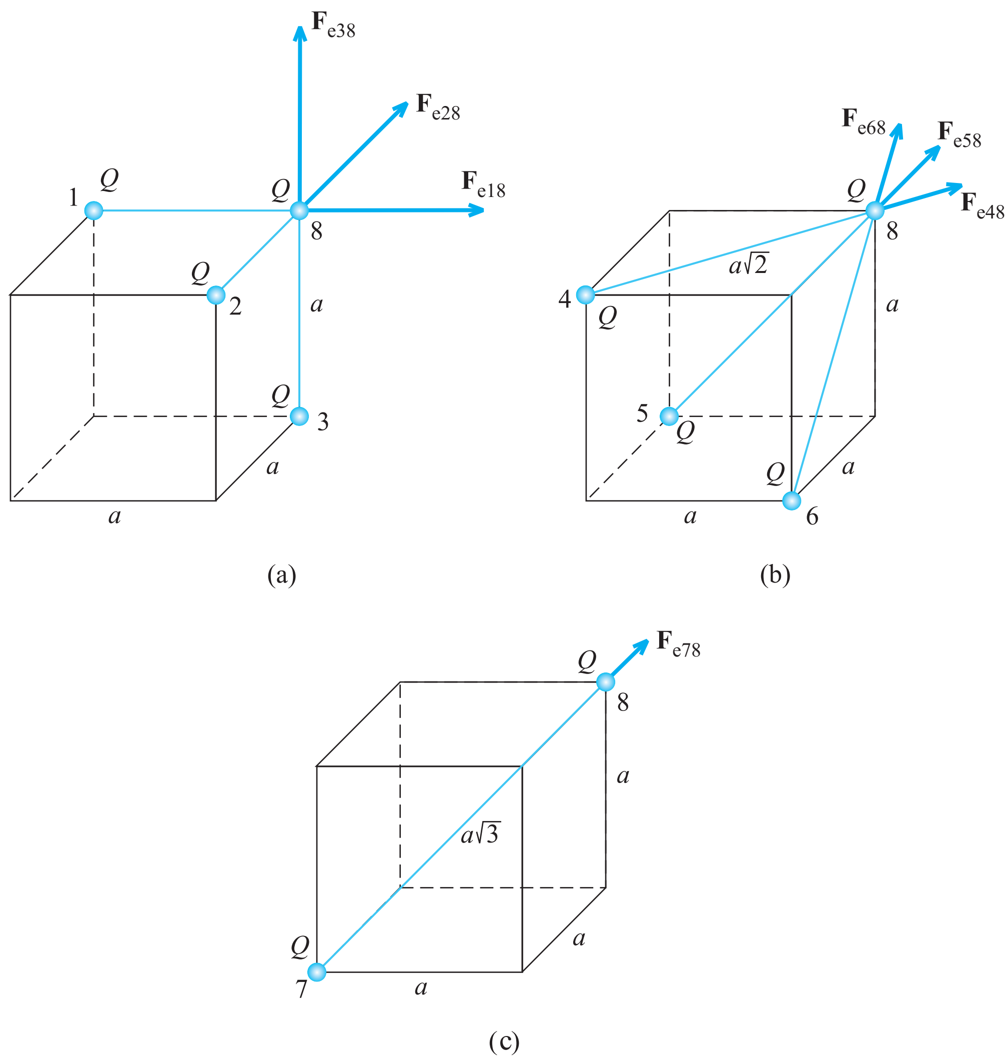

Figure P1.5 Evaluation of the total electric force on one of the eight charges placed at the vertices of a cube.

$$\mathbf{F}_{e18} + \mathbf{F}_{e28} + \mathbf{F}_{e38} = F_{e18}\sqrt{3}\,\hat{\mathbf{r}} = \frac{\sqrt{3}Q^2}{4\pi\varepsilon_0 a^2}\,\hat{\mathbf{r}} \qquad \text{(P1.23)}$$

(the vector sum of three vectors of equal magnitudes that are orthogonal to each other equals $\sqrt{3}$ times the magnitude of each of the vectors, in the same way the space diagonal of the cube of edge length a equals $a\sqrt{3}$), with $\hat{\mathbf{r}}$ being the unit vector along the cube diagonal (containing charge 8) – the radial unit vector with respect to the cube center. Similarly, Fig.P1.5(b) tells us that

$$\mathbf{F}_{\mathrm{e}48} + \mathbf{F}_{\mathrm{e}58} + \mathbf{F}_{\mathrm{e}68} = 3F_{\mathrm{e}48}\cos\alpha\,\hat{\mathbf{r}} = 3\,\frac{Q^2}{4\pi\varepsilon_0(a\sqrt{2})^2}\sqrt{\frac{2}{3}}\,\hat{\mathbf{r}} = \frac{\sqrt{6}Q^2}{8\pi\varepsilon_0 a^2}\,\hat{\mathbf{r}}$$

$$\left(\cos\alpha = \frac{\text{square diagonal}}{\text{cube diagonal}} = \frac{a\sqrt{2}}{a\sqrt{3}} = \sqrt{\frac{2}{3}}\right). \qquad \text{(P1.24)}$$

Note that this result can alternatively be obtained using the solution to Example 1.4, since vertices 4, 5, 6, and 8 of the cube in Fig.P1.5(b) form a regular tetrahedron (with the edge length $a\sqrt{2}$). Finally, from Fig.P1.5(c),

$$\mathbf{F}_{\mathrm{e}78} = \frac{Q^2}{4\pi\varepsilon_0(a\sqrt{3})^2}\,\hat{\mathbf{r}} = \frac{Q^2}{12\pi\varepsilon_0 a^2}\,\hat{\mathbf{r}}\,, \qquad \text{(P1.25)}$$

and hence the total electric force on charge 8

$$\mathbf{F}_{\mathrm{e}8} = \sum_{i=1}^{7}\mathbf{F}_{\mathrm{e}i8} = \frac{Q^2}{4\pi\varepsilon_0 a^2}\left(\sqrt{3} + \frac{\sqrt{6}}{2} + \frac{1}{3}\right)\hat{\mathbf{r}}\,. \qquad \text{(P1.26)}$$

Section 1.2 Definition of the Electric Field Intensity Vector

PROBLEM 1.8 Electric field due to three point charges in space. (a) Using Eq.(1.25), the electric field intensity vector at the coordinate origin (point O) in Fig.1.5(a), produced by the three point charges, Q_1, Q_2, and Q_3, which all appear to be at a distance $a = 1$ m from the observation (field) point, is given by

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1}{a^2}(-\hat{\mathbf{x}}) + \frac{Q_2}{a^2}(-\hat{\mathbf{y}}) + \frac{Q_3}{a^2}(-\hat{\mathbf{z}})\right]$$
$$= 9\,(-\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}} - 2\,\hat{\mathbf{z}})\ \mathrm{kV/m} \quad (Q_2 < 0)\,. \qquad \text{(P1.27)}$$

Its magnitude is $E = 9\sqrt{1 + 2^2 + 2^2}$ kV/m $= 27$ kV/m. The direction of $\mathbf{E}$ is determined by the unit vector $\hat{\mathbf{u}}_E = -(1/3)\,\hat{\mathbf{x}} + (2/3)\,\hat{\mathbf{y}} - (2/3)\,\hat{\mathbf{z}}$, based on which, we can, for instance, obtain the angles that $\mathbf{E}$ makes with the coordinate axes in Fig.1.5(a).

(b) For the observation point defined by $z = 100$ m at the z-axis, since $z \gg R$, with $R = \sqrt{2}$ m being the distance between any two of the charges Q_1, Q_2, and Q_3 in Fig.1.5(a), the system of these three charges, considered as a single charged object, is very small compared to the distance of the observation point from the center of the system. Hence, the system can be treated as a single point charge amounting to $Q_{\mathrm{tot}} = Q_1 + Q_2 + Q_3 = 1\ \mu$C. Because $z \gg R$, we can place Q_{tot} at the coordinate origin in Fig.1.5(a), instead of the exact center of the system of three charges, and Eq.(1.24) then gives the following for the electric field vector due to the three charges at this observation point:

$$\mathbf{E} \approx \frac{Q_{\mathrm{tot}}}{4\pi\varepsilon_0 z^2}\,\hat{\mathbf{z}} = 0.9\,\hat{\mathbf{z}}\ \mathrm{V/m} \quad (z \gg R)\,. \qquad \text{(P1.28)}$$

Section 1.3 Continuous Charge Distributions

PROBLEM 1.9 **Nonuniform volume charge in a cylinder.** We perform a similar volume integration as in Fig.1.9 and Eq.(1.34) but adopting $\mathrm{d}v$ as the volume of a thin cylindrical shell of radius r, thickness $\mathrm{d}r$, and finite length l, with r being the radial distance from the cylinder axis ($0 \le r \le a$). To compute $\mathrm{d}v$, the cylindrical shell can be flattened into a thin rectangular slab, with edges equal to $2\pi r$ (circumference of the shell), l, and $\mathrm{d}r$. The charge on the length l of the cylinder thus amounts to

$$Q = \int_v \rho \,\mathrm{d}v = \int_{r=0}^{a} \underbrace{\rho_0 \left(1 - \frac{r}{a}\right)}_{\rho} \underbrace{2\pi r \,\mathrm{d}r\, l}_{\mathrm{d}v} = \frac{\pi \rho_0 a^2 l}{3} \,. \tag{P1.29}$$

Using Eq.(1.31), the charge per unit length of the cylinder is

$$Q' = \frac{Q}{l} = \frac{\pi \rho_0 a^2}{3} \,. \tag{P1.30}$$

PROBLEM 1.10 **Nonuniform volume charge in a cube.** Adopting $\mathrm{d}v$ in the form of a slice of the cube with thickness $\mathrm{d}x$, the total charge of the cube turns out to be

$$Q = \int_v \rho \,\mathrm{d}v = \int_{x=0}^{a} \rho_0 \sin\left(\frac{\pi}{a}\, x\right) \underbrace{a^2 \,\mathrm{d}x}_{\mathrm{d}v} = \frac{2\rho_0 a^3}{\pi} \,. \tag{P1.31}$$

PROBLEM 1.11 **Nonuniform surface charge on a disk.** From Eqs.(1.29), the total charge of the disk is given by

$$Q = \int_S \rho_\mathrm{s} \,\mathrm{d}S = \int_{r=0}^{a} \underbrace{\rho_{\mathrm{s}0} \frac{r^2}{a^2}}_{\rho_\mathrm{s}} \underbrace{2\pi r \,\mathrm{d}r}_{\mathrm{d}S} = \frac{\pi \rho_{\mathrm{s}0} a^2}{2} \,, \tag{P1.32}$$

where $\mathrm{d}S$ is the surface area of an elemental ring of radius r ($0 \le r \le a$) and width $\mathrm{d}r$, computed as that of a thin strip of length equal to the ring circumference, $2\pi r$, and width $\mathrm{d}r$ (the ring can be straightened into a strip for this computation).

PROBLEM 1.12 **Nonuniform line charge along a rod.** Using the last expression in Eqs.(1.29) and the fact that $\mathrm{d}l = \mathrm{d}x$, the total charge of the rod comes out to be

$$Q = \int_l Q' \,\mathrm{d}l = \int_{x=0}^{l} Q'_0 \left[1 - \sin\left(\frac{\pi}{l}\, x\right)\right] \mathrm{d}x = Q'_0 l \left(1 - \frac{2}{\pi}\right) . \tag{P1.33}$$

Section 1.5 Electric Field Intensity Vector Due to Given Charge Distributions

PROBLEM 1.13 Field maximum at the axis of a ring. From Eq.(1.44), the algebraic intensity of the electric field vector (due to the charged ring) along the z-axis in Fig.1.11 is given by

$$E_z(z) = \frac{Qz}{4\pi\varepsilon_0 \left(z^2 + a^2\right)^{3/2}} \qquad (Q > 0) \ . \tag{P1.34}$$

We perform a standard procedure of equating to zero the derivative of $E_z(z)$ with respect to z, which yields

$$\frac{\mathrm{d}E_z}{\mathrm{d}z} = 0 \quad \longrightarrow \quad \frac{Q}{4\pi\varepsilon_0} \frac{\left(z^2 + a^2\right)^{3/2} - z(3/2)\left(z^2 + a^2\right)^{1/2} 2z}{\left(z^2 + a^2\right)^3} = 0$$
$$\longrightarrow \quad z^2 = \frac{a^2}{2} \quad \longrightarrow \quad z = \pm \frac{a}{\sqrt{2}} \ , \tag{P1.35}$$

where $z = a/\sqrt{2}$ corresponds to the maximum of the function $E_z(z)$ in the domain where it is positive (for $0 < z < \infty$), while at $z = -a/\sqrt{2}$ the negative function $E_z(z)$ (for $-\infty < z < 0$) is minimum, as shown in Fig.P1.6. However, at both locations the field intensity $|E_z(z)|$ reaches its maximum value, $E_{\max} = \sqrt{3}Q/(18\pi\varepsilon_0 a^2)$.

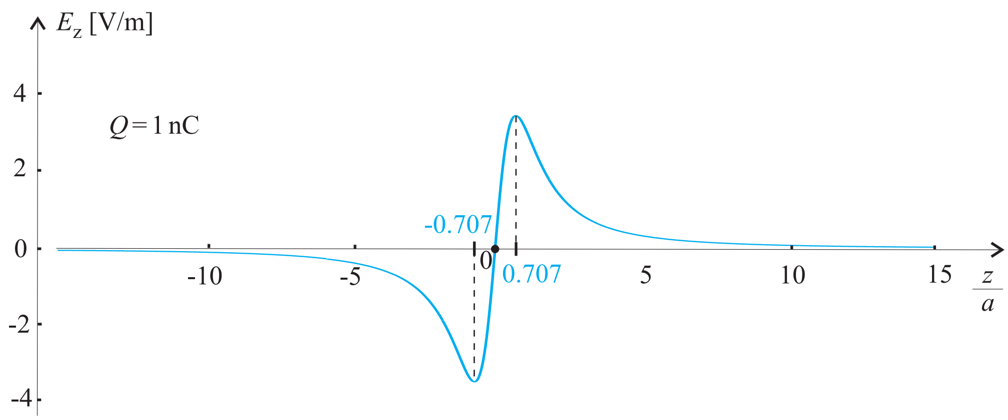

Figure P1.6 Plot of the algebraic intensity of the field vector **E** along the axis of the charged ring (z-axis) in Fig.1.11.

PROBLEM 1.14 Point charge equivalent to a charged semicircle. Using Eq.(1.51), the expression for the electric field intensity vector due to the charged semicircle at an arbitrary point along the z-axis in Fig.1.12(a) can be written as

$$\mathbf{E} = \frac{Q}{2\pi\varepsilon_0 \left(z^2 + a^2\right)^{3/2}} \left(-\frac{a}{\pi}\,\hat{\mathbf{x}} + \frac{z}{2}\,\hat{\mathbf{z}}\right) , \tag{P1.36}$$

with $Q = Q'\pi a$ being the total charge of the semicircle. For $|z| \gg a$, $z^2 + a^2 \approx z^2$, $\sqrt{z^2+a^2} \approx |z|$, and $a/|z|^3 \ll 1/z^2$, with which Eq.(P1.36) becomes [also see Eq.(1.45)]

$$\mathbf{E} \approx \frac{Q}{2\pi\varepsilon_0|z|^3}\left(-\frac{a}{\pi}\,\hat{\mathbf{x}} + \frac{z}{2}\,\hat{\mathbf{z}}\right) = \frac{Q}{2\pi\varepsilon_0}\left(-\frac{a}{\pi|z|^3}\,\hat{\mathbf{x}} + \frac{1}{2z^2}\frac{z}{|z|}\,\hat{\mathbf{z}}\right) \approx \frac{Q}{4\pi\varepsilon_0 z^2}\frac{z}{|z|}\,\hat{\mathbf{z}} \quad (|z| \gg a)\,, \tag{P1.37}$$

where $z/|z| = 1$ for $z > 0$ and $z/|z| = -1$ for $z < 0$. So, indeed, as long as the electric field at a very distant location along the z-axis in Fig.1.12(a), in both positive and negative directions, is concerned, the semicircular line charge is equivalent to a point charge Q placed at the coordinate origin.

PROBLEM 1.15 **Charged contour of complex shape.** Because of symmetry, the electric field vectors at the point O due to the two linear parts of the contour in Fig.1.51 cancel each other, $\mathbf{E}_2 + \mathbf{E}_4 = 0$, as indicated in Fig.P1.7. By means of Eq.(1.51), on the other side, the field vectors at the same point due to the semicircular parts of radii a and b are given, respectively, by

$$\mathbf{E}_1 = \frac{Q'}{2\pi\varepsilon_0 a}\,\hat{\mathbf{x}} \quad \text{and} \quad \mathbf{E}_3 = -\frac{Q'}{2\pi\varepsilon_0 b}\,\hat{\mathbf{x}} \quad \left[Q' = \frac{Q}{\pi(a+b)+2(b-a)}\right], \tag{P1.38}$$

for the x-axis in Fig.P1.7, where Q' stands for the line charge density of the contour. The total electric field is thus

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_3 = \frac{Q(b-a)}{2\pi\varepsilon_0 ab\,[\pi(a+b)+2(b-a)]}\,\hat{\mathbf{x}}\,. \tag{P1.39}$$

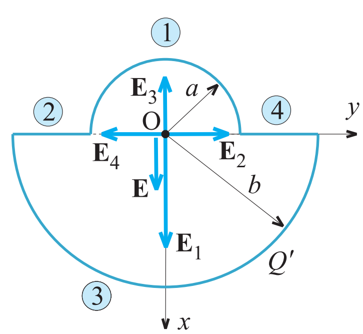

Figure P1.7 Computing the electric field at the center (point O) of the uniformly charged contour in Fig.1.51.

PROBLEM 1.16 **Nonuniform line charge along a semicircle.** (a) Since $Q'(\phi)$ is an odd (sine) function of ϕ within the symmetric integration limits, $-\pi/2$ and $\pi/2$, the total charge of the semicircle is zero. Namely, as $\mathrm{d}l = a\,\mathrm{d}\phi$ in Fig.1.12(a),

$$Q = \int_l Q'(\phi)\,\mathrm{d}l = Q'_0\int_{\phi=-\pi/2}^{\pi/2}\sin\phi\;\mathrm{d}\phi = 0\,. \tag{P1.40}$$

(b) Modifying Eqs.(1.48)-(1.50), we obtain

$$E_x = -\frac{a^2}{4\pi\varepsilon_0 R^3}\int_{-\pi/2}^{\pi/2} Q'(\phi)\cos\phi \, \mathrm{d}\phi = -\frac{Q_0' a^2}{4\pi\varepsilon_0 R^3}\int_{-\pi/2}^{\pi/2} \sin\phi\cos\phi \, \mathrm{d}\phi = 0 \quad \text{(P1.41)}$$

(the integrand, $\sin\phi\cos\phi$, is a product of an odd and an even function in ϕ, which results in an odd function, and hence the integral is zero),

$$E_y = -\frac{Q_0' a^2}{4\pi\varepsilon_0 R^3}\int_{-\pi/2}^{\pi/2} \sin^2\phi \, \mathrm{d}\phi = -\frac{Q_0' a^2}{4\pi\varepsilon_0 R^3}\int_{-\pi/2}^{\pi/2} \frac{1-\cos 2\phi}{2} \, \mathrm{d}\phi$$

$$= -\frac{Q_0' a^2}{8\pi\varepsilon_0 R^3}\left(\int_{-\pi/2}^{\pi/2} \mathrm{d}\phi - \int_{-\pi/2}^{\pi/2} \cos 2\phi \, \mathrm{d}\phi\right) = -\frac{Q_0' a^2}{8\varepsilon_0 R^3} \qquad \left(R = \sqrt{z^2+a^2}\right), \quad \text{(P1.42)}$$

$$E_z = \frac{az}{4\pi\varepsilon_0 R^3}\int_{-\pi/2}^{\pi/2} Q'(\phi) \, \mathrm{d}\phi = 0 \, , \quad \text{(P1.43)}$$

so that the electric field vector at the point P in Fig.1.12(a) comes out to be

$$\mathbf{E} = -\frac{Q_0' a^2}{8\varepsilon_0 \left(z^2+a^2\right)^{3/2}} \, \hat{\mathbf{y}} \, . \quad \text{(P1.44)}$$

PROBLEM 1.17 **Line charge along three-quarters of a circle.** Let us extend the semicircle in Fig.1.12(a) to a three-quarter circle such that the line charge exists along the circle (of radius a) for $-\pi \le \phi \le \pi/2$. Then, using Eqs.(1.48)-(1.50) with $z = 0$ (the electric field is calculated at the arc center) and modified integration limits, we have

$$E_x = -\frac{Q'}{4\pi\varepsilon_0 a}\int_{\phi=-\pi}^{\pi/2} \cos\phi \, \mathrm{d}\phi = -\frac{Q'}{4\pi\varepsilon_0 a} \, , \quad E_y = -\frac{Q'}{4\pi\varepsilon_0 a}\int_{-\pi}^{\pi/2} \sin\phi \, \mathrm{d}\phi$$

$$= \frac{Q'}{4\pi\varepsilon_0 a} \, , \quad \text{and} \quad E_z = 0 \quad \longrightarrow \quad \mathbf{E} = \frac{Q}{6\pi\varepsilon_0 a^2}\left(-\hat{\mathbf{x}} + \hat{\mathbf{y}}\right) , \quad \text{(P1.45)}$$

where $Q' = 2Q/(3\pi a)$ is the line charge density along the arc (the arc length is $l = 3\pi a/2$).

PROBLEM 1.18 **Line charge along a quarter of a circle.** This case differs from that in Fig.1.12(a) and Eqs.(1.48)-(1.50) only in the integration limits. If we place the quarter-circle such that it extends (along the circle of radius a) in the range $0 \le \phi \le \pi/2$ in Fig.1.12(a), the electric field components due to this arc at an arbitrary point along the z-axis are given by

$$E_x = -\frac{Q' a^2}{4\pi\varepsilon_0 R^3}\int_{\phi=0}^{\pi/2} \cos\phi \, \mathrm{d}\phi = -\frac{Q' a^2}{4\pi\varepsilon_0 R^3} \, , \quad E_y = -\frac{Q' a^2}{4\pi\varepsilon_0 R^3}\int_{0}^{\pi/2} \sin\phi \, \mathrm{d}\phi$$

$$= -\frac{Q' a^2}{4\pi\varepsilon_0 R^3} \, , \quad E_z = \frac{Q' az}{4\pi\varepsilon_0 R^3}\int_{0}^{\pi/2} \mathrm{d}\phi = \frac{Q' az}{8\varepsilon_0 R^3} \, , \quad \text{(P1.46)}$$

and the field vector is

$$\mathbf{E} = \frac{Q'a}{4\varepsilon_0\left(z^2+a^2\right)^{3/2}}\left(-\frac{a}{\pi}\,\hat{\mathbf{x}} - \frac{a}{\pi}\,\hat{\mathbf{y}} + \frac{z}{2}\,\hat{\mathbf{z}}\right). \tag{P1.47}$$

PROBLEM 1.19 **Semi-infinite line charge.** We refer to Fig.P1.8, in place of Fig.1.13, and use Eq.(1.56) with $Q/l = Q'$, $d = |y|$, $\hat{\mathbf{x}} = \hat{\mathbf{y}}$, $\hat{\mathbf{z}} = \hat{\mathbf{x}}$, $\theta_1 \to -\pi/2$, $\sin\theta_2 = -x/\sqrt{x^2+y^2}$, and $\cos\theta_2 = |y|/\sqrt{x^2+y^2}$ to obtain the electric field intensity vector at an arbitrary point (x, y) in the xy-plane:

$$\mathbf{E} = \frac{Q'}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2}}\,\hat{\mathbf{x}} + \frac{1}{y}\left(1 - \frac{x}{\sqrt{x^2+y^2}}\right)\hat{\mathbf{y}}\right], \tag{P1.48}$$

where y appears instead of $|y|$ in the expression for the y-component of $\mathbf{E}$ because it properly, together with $\hat{\mathbf{y}}$, determines the actual direction of that component for $y < 0$, as $-\hat{\mathbf{y}}/|y|$ in place of $\hat{\mathbf{y}}/|y|$.

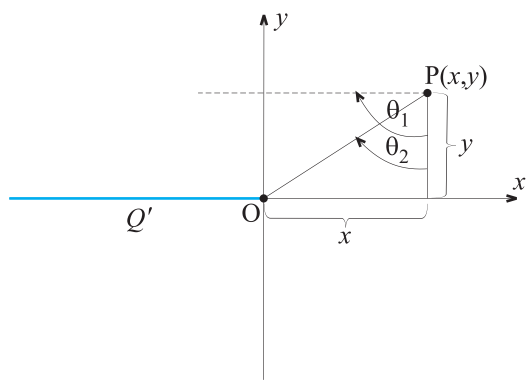

Figure P1.8 Computing the electric field due to a semi-infinite line charge.

PROBLEM 1.20 **Half-positive, half-negative infinite line charge.** Using the expression for the electric field due to a semi-infinite line charge from the previous problem, with $x = 0$ and $y = d$, the electric field vector at the point $(0, d)$ due to the positive semi-infinite line charge (extending along the negative part of the x-axis) comes out to be

$$\mathbf{E}_{Q'} = \frac{Q'}{4\pi\varepsilon_0 d}\left(\hat{\mathbf{x}} + \hat{\mathbf{y}}\right). \tag{P1.49}$$

Note that this result can also be obtained by means of Eq.(1.56) with $\theta_1 \to -\pi/2$ and $\theta_2 = 0$, along with other substitutions. Because of symmetry, the field vector due to the negative semi-infinite line charge (extending for $0 < x < \infty$) equals

$$\mathbf{E}_{-Q'} = \frac{Q'}{4\pi\varepsilon_0 d}\left(\hat{\mathbf{x}} - \hat{\mathbf{y}}\right), \tag{P1.50}$$

and hence the total field

$$\mathbf{E} = \mathbf{E}_{Q'} + \mathbf{E}_{-Q'} = \frac{Q'}{2\pi\varepsilon_0 d}\,\hat{\mathbf{x}}\,. \tag{P1.51}$$

PROBLEM 1.21 **Charged square contour.** The electric field intensity at the point P in Fig.P1.9 due to the charge along one of the square sides, E_1, is given by Eq.(1.56) with $d = h = \sqrt{3}a/2$ (height of an equilateral triangle of side length a), $\theta_1 = -\pi/6$, and $\theta_2 = \pi/6$, so that

$$E_1 = \frac{Q'}{4\pi\varepsilon_0\sqrt{3}a/2}\left[\sin\frac{\pi}{6} - \sin\left(-\frac{\pi}{6}\right)\right] = \frac{Q'}{2\sqrt{3}\pi\varepsilon_0 a}\,. \tag{P1.52}$$

The total electric field vector at the point P is thus (Fig.P1.9)

$$\mathbf{E} = 4E_1\cos\alpha\,\hat{\mathbf{z}} = 4E_1\sqrt{1-\sin^2\alpha}\,\hat{\mathbf{z}} = \frac{2\sqrt{2}Q'}{3\pi\varepsilon_0 a}\,\hat{\mathbf{z}}\,, \quad \text{where} \quad \sin\alpha = \frac{a/2}{h} = \frac{\sqrt{3}}{3}\,. \tag{P1.53}$$

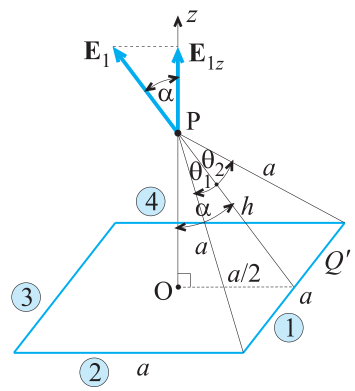

Figure P1.9 Evaluation of the electric field along the axis of a charged square contour.

PROBLEM 1.22 **Point charge equivalent to a charged disk.** Assume that $z > 0$ and let $z \gg a$ in Fig.1.14. Hence, $\sqrt{z^2 + a^2} \approx z$ and Eq.(1.63) becomes

$$\mathbf{E} = \frac{\rho_s}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)\hat{\mathbf{z}} \approx \frac{\rho_s}{2\varepsilon_0}\left(1 - \frac{z}{z}\right)\hat{\mathbf{z}} = 0\,, \tag{P1.54}$$

which, of course, is too crude of an approximation. Rather, we first perform a rationalization of the resulting fraction as follows:

$$E = \frac{\rho_s}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right) = \frac{\rho_s}{2\varepsilon_0}\frac{\sqrt{a^2+z^2}-z}{\sqrt{a^2+z^2}} = \frac{\rho_s}{2\varepsilon_0}\frac{\sqrt{a^2+z^2}-z}{\sqrt{a^2+z^2}}\frac{\sqrt{a^2+z^2}+z}{\sqrt{a^2+z^2}+z}$$
$$= \frac{\rho_s}{2\varepsilon_0}\frac{(a^2+z^2)-z^2}{\sqrt{a^2+z^2}(\sqrt{a^2+z^2}+z)} \approx \frac{\rho_s}{2\varepsilon_0}\frac{a^2}{z(z+z)} = \frac{\rho_s\pi a^2}{4\pi\varepsilon_0 z^2} = \frac{Q}{4\pi\varepsilon_0 z^2}\,, \tag{P1.55}$$

which indeed is the field of a point charge $Q = \rho_{\rm s}\pi a^2$ placed at the disk center. The case $z < 0$ and $|z| = -z \gg a$ can be analyzed in a completely analogous manner.

PROBLEM 1.23 **Field due to a nonuniformly charged disk.** Using the same notation as in Fig.1.14, Eq.(1.63) is now modified as follows:

$$\mathbf{E} = \frac{z}{2\varepsilon_0}\int_{r=0}^{a} \rho_{\rm s}(r)\,\frac{{\rm d}R}{R^2}\,\hat{\mathbf{z}} = \frac{\rho_{\rm s0}z}{2\varepsilon_0 a^2}\int_{r=0}^{a} r^2\,\frac{{\rm d}R}{R^2}\,\hat{\mathbf{z}} = \frac{\rho_{\rm s0}z}{2\varepsilon_0 a^2}\int_{r=0}^{a}(R^2 - z^2)\,\frac{{\rm d}R}{R^2}\,\hat{\mathbf{z}}$$

$$= \frac{\rho_{\rm s0}z}{2\varepsilon_0 a^2}\left(\int_{r=0}^{a}{\rm d}R - z^2\int_{r=0}^{a}\frac{{\rm d}R}{R^2}\right)\hat{\mathbf{z}} = \frac{\rho_{\rm s0}z}{2\varepsilon_0 a^2}\left[R\Big|_{r=0}^{a} - z^2\left(-\frac{1}{R}\right)\Big|_{r=0}^{a}\right]\hat{\mathbf{z}}$$

$$= \frac{\rho_{\rm s0}z}{2\varepsilon_0 a^2}\left(\sqrt{a^2+z^2} - |z| + \frac{z^2}{\sqrt{a^2+z^2}} - \frac{z^2}{|z|}\right)\hat{\mathbf{z}}\,. \qquad \text{(P1.56)}$$

Note that in the case of a charged isolated metallic disk, the charge distribution considered here is physically much more realistic than the one (uniform) in Example 1.10.

PROBLEM 1.24 **Nonuniformly charged spherical surface.** (a) As the charge density function of the sphere is antisymmetrical with respect to the equatorial plane (plane $\theta = \pi/2$), positive for $0 < \theta < \pi/2$ and negative, with the same "waveform," for $\pi/2 < \theta < \pi$, the total charge of the sphere is zero. Alternatively, this can be shown referring to Fig.1.16 and using Eqs.(1.29) and (1.65), which result in the following integral for the total charge:

$$Q = \int_S \rho_{\rm s}(\theta)\,{\rm d}S = \int_{\theta=0}^{\pi} \rho_{\rm s0}\sin 2\theta\, 2\pi a^2 \sin\theta\,{\rm d}\theta = 0 \qquad \text{(P1.57)}$$

(a product of an antisymmetrical ($\sin 2\theta$) and a symmetrical ($\sin\theta$) function in θ with respect to the equatorial plane is an antisymmetrical function, and its integral from 0 to π is zero).

(b) Because of the antisymmetry of the charge distribution of the sphere, the contribution of its upper half (for $0 \le \theta \le \pi/2$) to the electric field vector ($\mathbf{E}$) at the sphere center (point O in Fig.1.16) is the same as the contribution of the lower half (for $\pi/2 \le \theta \le \pi$), and hence the resulting field integral from 0 to π equals twice the integral from 0 to $\pi/2$ (over the upper hemisphere). Therefore, Eq.(1.67) gives

$$\mathbf{E} = -\frac{1}{2\varepsilon_0}\,2\int_0^{\pi/2}\rho_{\rm s}(\theta)\sin\theta\cos\theta\,{\rm d}\theta\,\hat{\mathbf{z}} = -\frac{\rho_{\rm s0}}{2\varepsilon_0}\int_0^{\pi/2}\sin^2 2\theta\,{\rm d}\theta\,\hat{\mathbf{z}}$$

$$= -\frac{\rho_{\rm s0}}{2\varepsilon_0}\int_0^{\pi/2}\frac{1-\cos 4\theta}{2}\,{\rm d}\theta\,\hat{\mathbf{z}} = -\frac{\pi\rho_{\rm s0}}{8\varepsilon_0}\,\hat{\mathbf{z}}\,. \qquad \text{(P1.58)}$$

PROBLEM 1.25 **Infinite charged sheet with a circular hole.** (a) To compute the electric field intensity vector at an arbitrary point along the z-axis due to the infinite charged sheet with a circular hole as an integral of individual fields due to elementary rings (as in Fig.1.14) making it, from the periphery of the hole ($r = a$) to infinity ($r \to \infty$), we simply change the integration limits in Eq.(1.63), which yields

$$\mathbf{E} = \underbrace{\frac{\rho_s z}{2\varepsilon_0} \int_{r=a}^{\infty} \frac{\mathrm{d}R}{R^2}\, \hat{\mathbf{z}} = \frac{\rho_s z}{2\varepsilon_0} \left(-\frac{1}{R} \right) \Bigg|_{r=a}^{\infty}}_{\text{integrating } \mathrm{d}\mathbf{E} \text{ due to elementary rings}} \hat{\mathbf{z}} = \frac{\rho_s z}{2\varepsilon_0 \sqrt{a^2 + z^2}}\, \hat{\mathbf{z}} \qquad (-\infty < z < \infty)\,. \tag{P1.59}$$

(b) Alternatively, we can represent the charge distribution of the infinite sheet with the hole as a sum of charge distributions of an infinite sheet of continuous charge with density ρ_s (e.g., positively charged, assuming $\rho_s > 0$) and a disk of charge with density $-\rho_s$ (negatively charged) and the same dimensions as the hole. By the superposition principle, the electric field vector of the original (resultant) charge distribution can then be obtained using Eqs.(1.64) and (1.63) (see Figs.1.15 and 1.14) as

$$\mathbf{E} = \underbrace{\frac{\rho_s}{2\varepsilon_0} \frac{z}{|z|}\, \hat{\mathbf{z}}}_{\text{positive sheet}} + \underbrace{\frac{-\rho_s}{2\varepsilon_0} \left(\frac{z}{|z|} - \frac{z}{\sqrt{a^2 + z^2}} \right) \hat{\mathbf{z}}}_{\text{negative disk}} = \underbrace{\frac{\rho_s z}{2\varepsilon_0 \sqrt{a^2 + z^2}}\, \hat{\mathbf{z}}}_{\text{sheet with hole}}\,, \tag{P1.60}$$

and this, of course, is the same result as in Eq.(P1.59).

PROBLEM 1.26 Force on a charged semicylinder due to a line charge. By virtue of Newton's third law, the law of action and reaction, the per-unit-length electric force on the charged semicylinder due to the line charge in Fig.1.17(a), $\mathbf{F}'_e$, is equal in magnitude and opposite in direction to that on the line charge due the semicylinder, i.e., that given in Eq.(1.71), and hence (Fig.P1.10)

$$\mathbf{F}'_e = -\frac{Q' \rho_s}{\pi \varepsilon_0}\, \hat{\mathbf{x}}\,. \tag{P1.61}$$

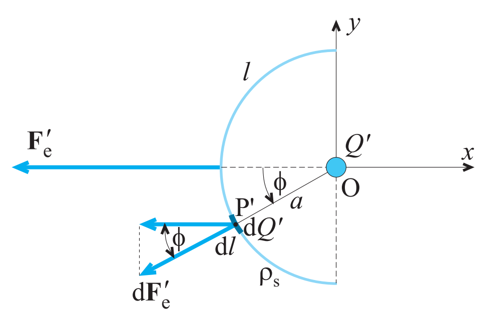

Figure P1.10 Evaluation [reverse to that in Fig.1.17(b)] of the per-unit-length electric force ($\mathbf{F}'_e$) on a charged semicylinder due to a line charge (of density Q') positioned along the semicylinder axis.

Alternatively, we can obtain the same result by summing (integrating) per-unit-length electric forces on individual elemental strips, of width $\mathrm{d}l$ and equivalent line charge density $\mathrm{d}Q'$ (found in Example 1.13), constituting the semicylinder

[Fig.1.17(b)]. The magnitudes of these forces are

$$\mathrm{d}F'_\mathrm{e} = \mathrm{d}Q'E\,, \quad \text{where} \quad \mathrm{d}Q' = \rho_\mathrm{s}\,\mathrm{d}l = \rho_\mathrm{s}a\,\mathrm{d}\phi \quad \text{and} \quad E = \frac{Q'}{2\pi\varepsilon_0 a}\,, \qquad \text{(P1.62)}$$

with E standing for the electric field intensity vector due to the line charge of density Q' computed at the location of the elemental strip. Thus, as shown in Fig.P1.10 [also see Eq.(1.70)],

$$\mathbf{F}'_\mathrm{e} = \int_l \mathrm{d}\mathbf{F}'_\mathrm{e} = \int_l \mathrm{d}F'_\mathrm{e}\cos\phi\,(-\,\hat{\mathbf{x}}) = -\frac{Q'\rho_\mathrm{s}}{2\pi\varepsilon_0}\int_{\phi=-\pi/2}^{\pi/2}\cos\phi\;\mathrm{d}\phi\;\hat{\mathbf{x}} = -\frac{Q'\rho_\mathrm{s}}{\pi\varepsilon_0}\,\hat{\mathbf{x}}\,. \qquad \text{(P1.63)}$$

PROBLEM 1.27 **Charged strip.** As this is a two-dimensional problem, we solve it in a cross-sectional plane. Fig.P1.11 shows a cross section of the strip and an arbitrary field point P, with the geometry of the problem being defined by angles θ_1 and θ_2, and the perpendicular distance d of P from the plane of the strip. Note the similarity of the figure with that in Fig.1.13, although the two geometries are very different. We subdivide the structure into differentially narrow strips of width $\mathrm{d}l = \mathrm{d}y$, and compute the associated elementary electric fields at the point P as in Example 1.13:

$$\mathrm{d}E = \frac{\rho_\mathrm{s}\,\mathrm{d}y}{2\pi\varepsilon_0 R} \qquad \left(R = \sqrt{y^2 + d^2}\right). \qquad \text{(P1.64)}$$

We break the vector $\mathrm{d}\mathbf{E}$ in Fig.P1.11 onto its x- and y-components (suitable for integration),

$$\mathrm{d}E_x = \mathrm{d}E\cos\theta\,, \qquad \mathrm{d}E_y = -\,\mathrm{d}E\sin\theta\,, \qquad \text{(P1.65)}$$

which are then integrated along the width of the charged strip,

$$E_x = \frac{\rho_\mathrm{s}}{2\pi\varepsilon_0}\int_{y=y_1}^{y_2}\frac{\cos\theta\,\mathrm{d}y}{R}\,, \qquad E_y = -\frac{\rho_\mathrm{s}}{2\pi\varepsilon_0}\int_{y_1}^{y_2}\frac{\sin\theta\,\mathrm{d}y}{R}\,. \qquad \text{(P1.66)}$$

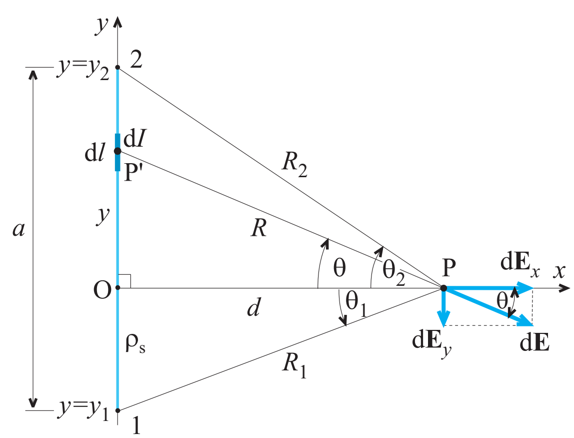

Figure P1.11 Evaluation of the electric field at an arbitrary point in space (P) due to an infinitely long uniformly charged strip of width a (cross-sectional view).

The integration is performed invoking the relationships in Eqs.(4.43) and (4.44) (in Chapter 4) to change the integration variable from y to θ, as follows [also see Eqs.(4.45) and (4.46)]:

$$E_x = \frac{\rho_s}{2\pi\varepsilon_0}\int_{\theta=\theta_1}^{\theta_2} d\theta = \frac{\rho_s}{2\pi\varepsilon_0}(\theta_2-\theta_1)\,, \quad E_y = -\frac{\rho_s}{2\pi\varepsilon_0}\int_{\theta_1}^{\theta_2}\frac{\sin\theta}{\cos\theta}\,d\theta$$

$$= \frac{\rho_s}{2\pi\varepsilon_0}\int_{\theta_1}^{\theta_2}\frac{d(\cos\theta)}{\cos\theta} = \frac{\rho_s}{2\pi\varepsilon_0}\ln\frac{\cos\theta_2}{\cos\theta_1} = \frac{\rho_s}{2\pi\varepsilon_0}\ln\frac{R_1}{R_2}\,, \quad \text{(P1.67)}$$

where R_1 and R_2 stand for the distances of the point P from the starting and ending points, respectively, of the line representing the cross section of the charged strip in Fig.P1.11. The electric field vector at this point is hence

$$\mathbf{E} = \frac{\rho_s}{2\pi\varepsilon_0}\left[(\theta_2-\theta_1)\,\hat{\mathbf{x}} + \hat{\mathbf{y}}\ln\frac{R_1}{R_2}\right]. \quad \text{(P1.68)}$$

Note that $\mathbf{E}$ can also be expressed using the charge per unit length of the strip, $Q' = \rho_s a$.

PROBLEM 1.28 **Two parallel oppositely charged strips.** The electric field intensity at the point A due to the charge of one of the strips, E_1, is given by $\mathbf{E} = \rho_s[(\theta_2-\theta_1)\,\hat{\mathbf{x}} + \hat{\mathbf{y}}\ln(R_1/R_2)]/(2\pi\varepsilon_0)$ (previous problem) with $\theta_1 = -\pi/4$, $\theta_2 = \pi/4$, $R_1 = R_2\,(= \sqrt{2}a/2)$, as can be seen in Fig.P1.12. By means of the superposition principle, the magnitude of the total electric field vector amounts to

$$E = 2E_1 = \frac{\rho_s}{2\varepsilon_0}\,, \quad \text{(P1.69)}$$

with respect to the reference direction of $\mathbf{E}$ indicated in Fig.P1.12. Having in mind Eq.(1.64) and Fig.1.15, we note that this is exactly a half of the field intensity ($E = \rho_s/\varepsilon_0$) that would be obtained if the two strips were of infinite widths (infinite sheets of charge).

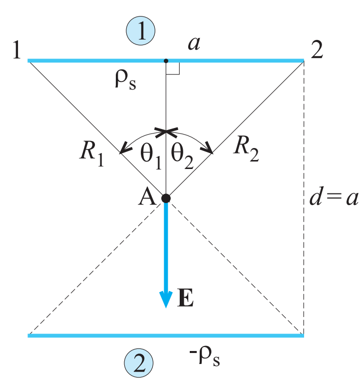

Figure P1.12 Computing the electric field (at the point A) due to the two parallel oppositely charged strips in Fig.1.52 (cross section of the structure).

Section 1.6 Definition of the Electric Scalar Potential

PROBLEM 1.29 Work in an electrostatic field. The work done by electric forces in moving the charge $Q = 1$ nC along the straight line joining the two points, O$(0,0,0)$ and P$(1\text{ m},1\text{ m},1\text{ m})$, equals the work computed along any other path between the points, so we choose the three-segment path shown in Fig.P1.13. Based on Eqs.(1.73) and (1.23), we thus have

$$W_\text{e} = \int_\text{O}^\text{P} \mathbf{F}_\text{e} \cdot \text{d}\mathbf{l} = Q \int_\text{O}^\text{P} \mathbf{E} \cdot \text{d}\mathbf{l} = Q \left(\int_{x=0}^{a} E_x \, \text{d}x + \int_{y=0}^{a} E_y \, \text{d}y + \int_{z=0}^{a} E_z \, \text{d}z \right)$$

$$= 10^{-9} \left(\int_0^1 x \, \text{d}x + \int_0^1 y^2 \, \text{d}y - \int_0^1 \text{d}z \right) \text{J} = \left(\frac{1}{2} + \frac{1}{3} - 1 \right) \text{nJ} = -166.7 \text{ pJ} , \tag{P1.70}$$

where $a = 1$ m and $\mathbf{E} \cdot \text{d}\mathbf{l} = \mathbf{E} \cdot \text{d}x \, \hat{\mathbf{x}} = E_x \, \hat{\mathbf{x}} \cdot \text{d}x \, \hat{\mathbf{x}} = E_x \, \text{d}x$ along the first segment (along the x-axis) of the integration path, and similarly for the other two segments.

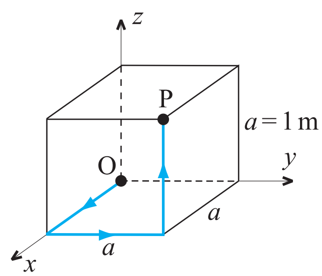

Figure P1.13 Integration path for finding the work of electric forces in moving the charge Q from the coordinate origin to the point P in the electrostatic field given by $\mathbf{E} = (x \, \hat{\mathbf{x}} + y^2 \, \hat{\mathbf{y}} - \hat{\mathbf{z}})$ V/m (x, y, z in m).

PROBLEM 1.30 Work in the field of a point charge. Using Eqs.(1.73) and (1.23), the work is given by

$$W_\text{e} = Q_2 \int_{\text{M}_1}^{\text{M}_2} \mathbf{E}_1 \cdot \text{d}\mathbf{l} , \tag{P1.71}$$

where $\mathbf{E}_1$ is the electric field vector due to the charge Q_1 in Fig.1.53. The lines of this field are radial with respect to the charge (the center of the square contour), and that is why we adopt a convenient integration path shown in Fig.P1.14, which consists of an arc of radius $a/2$ between points M_1 and A and a straight line segment between points A and M_2, and obtain [also see the similar integration in Fig.1.23 and Eq.(1.86)]

$$W_\text{e} = Q_2 \left(\int_{\text{M}_1}^{\text{A}} \mathbf{E}_1 \cdot \text{d}\mathbf{l} + \int_{\text{A}}^{\text{M}_2} \mathbf{E}_1 \cdot \text{d}\mathbf{l} \right) = Q_2 \int_{r=a/2}^{\sqrt{2}a/2} E_1(r) \, \text{d}r$$

$$= Q_2 \int_{a/2}^{\sqrt{2}a/2} \frac{Q_1}{4\pi\varepsilon_0 r^2}\,\mathrm{d}r = \frac{Q_1 Q_2}{4\pi\varepsilon_0}\left(\frac{1}{a/2} - \frac{1}{\sqrt{2}a/2}\right) = \frac{(2-\sqrt{2})Q_1 Q_2}{4\pi\varepsilon_0 a} = -526.5\ \mathrm{nJ}\,, \tag{P1.72}$$

where the line integral along the arc is zero because $\mathbf{E}$ is perpendicular to $\mathrm{d}\mathbf{l}$ during the integration.

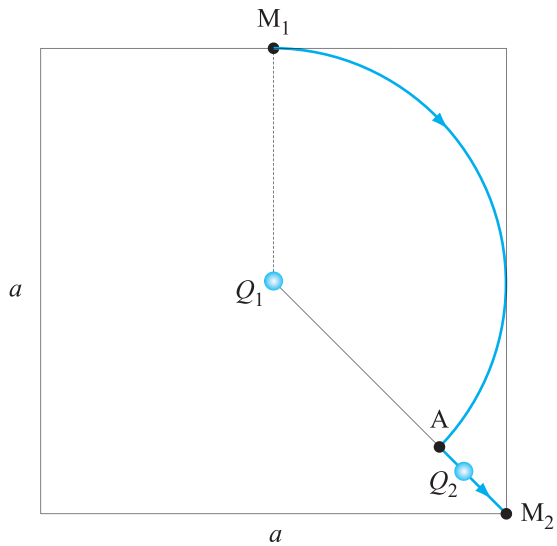

Figure P1.14 Integration path for finding the work done by electric forces in carrying the charge Q_2 between points M_1 and M_2 in the electric field due to the charge Q_1 in Fig.1.53.

Section 1.7 Electric Potential Due to Given Charge Distributions

PROBLEM 1.31 Electric potential due to three point charges in space. (a) Distances of the observation point $(0, 0, 2\,\mathrm{m})$ (at the z-axis) from the three point charges, Q_1, Q_2, and Q_3, in Fig.1.5(a) are, respectively, $R_1 = R_2 = \sqrt{1+2^2}$ m $= \sqrt{5}$ m and $R_3 = 1$ m. By means of Eq.(1.81), the electric potential at this point (with respect to the reference point at infinity) produced by the three charges amounts to

$$V = V_1 + V_2 + V_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{R_1} + \frac{Q_2}{R_2} + \frac{Q_3}{R_3}\right) = 13.95\ \mathrm{kV}\,. \tag{P1.73}$$

(b) For the observation point $(1\,\mathrm{m}, 1\,\mathrm{m}, 1\,\mathrm{m})$, $R_1 = R_2 = R_3 = \sqrt{2}$ m, which, for R_3, for instance, can easily be obtained by realizing that this point, its projections on xz and yz planes, and charge Q_3 form a square of edge length 1 m, or, more formally, as $\sqrt{(1-0)^2 + (1-0)^2 + (1-1)^2}$ m, based on Eq.(1.7). Hence, Eq.(P1.73) now gives $V = 6.35$ kV.

PROBLEM 1.32 **Point charge and an arbitrary reference point.** According to Fig.P1.15, we apply the same integration procedure as in Eq.(1.86) with only difference being the expression for the field intensity, $E = Q/(4\pi\varepsilon_0 x^2)$ in place of $E = Q'/(2\pi\varepsilon_0 x)$, which results in

$$V = \int_{\mathrm{P}}^{\mathrm{M}} \mathbf{E}\cdot\mathrm{d}\mathbf{l} + \int_{\mathrm{M}}^{\mathcal{R}} \mathbf{E}\cdot\mathrm{d}\mathbf{l} = \int_{x=r}^{r_{\mathcal{R}}} E\,\mathrm{d}x = \int_{r}^{r_{\mathcal{R}}} \frac{Q}{4\pi\varepsilon_0 x^2}\,\mathrm{d}x = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{r_{\mathcal{R}}}\right). \tag{P1.74}$$

Note that for $r_{\mathcal{R}} \to \infty$ (adopting a reference point at infinity), we, of course, obtain the corresponding expression for the potential in Eq.(1.80).

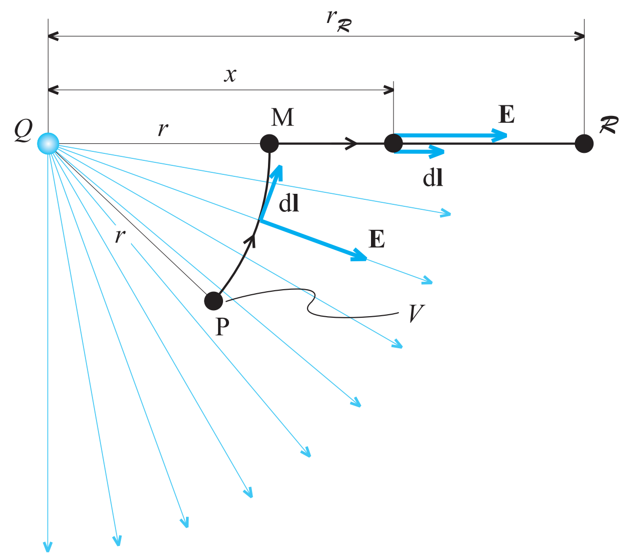

Figure P1.15 Convenient integration path (consisting of an arc between points P and M and a straight line segment between points M and $\mathcal{R}$) for the evaluation of the electric potential (at the point P) due to a point charge Q with respect to an arbitrary reference point $\mathcal{R}$.

PROBLEM 1.33 **Potential due to a semicircular line charge.** Similarly to the integration in Eq.(1.85), the potential at the point P in Fig.1.12 is found as

$$V = \frac{1}{4\pi\varepsilon_0}\int_l \frac{Q'\,\mathrm{d}l}{R} = \frac{Q'}{4\pi\varepsilon_0 R}\underbrace{\int_l \mathrm{d}l}_{\pi a} = \frac{Q'a}{4\varepsilon_0\sqrt{z^2+a^2}}\,. \tag{P1.75}$$

PROBLEM 1.34 **Potential due to a charged disk.** We use the same subdivision of the disk into elemental rings as in Fig.1.14 and integrate the electric potential due to these rings. Noting that the expression for the potential of a ring (of radius a) in Eq.(1.85) can be written as $V = Q/(4\pi\varepsilon_0 R)$, with $Q = Q'2\pi a$ being the total charge of the ring, the potential at the point P in Fig.1.14 due to the ring of radius r, width $\mathrm{d}r$, and charge $\mathrm{d}Q = \rho_\mathrm{s}\,\mathrm{d}S$ [Eq.(1.59)] equals

$$\mathrm{d}V = \frac{\mathrm{d}Q}{4\pi\varepsilon_0 R} = \frac{\rho_\mathrm{s} r\,\mathrm{d}r}{2\varepsilon_0 R}\,, \quad R = \sqrt{r^2+z^2}\,, \tag{P1.76}$$

where $\mathrm{d}S = 2\pi r\,\mathrm{d}r$ is the surface area of the ring [Eq.(1.60)]. A similar integration as in Eqs.(1.61)-(1.63) then gives

$$V = \int_S \mathrm{d}V = \frac{\rho_\mathrm{s}}{2\varepsilon_0}\int_{r=0}^{a}\frac{r\,\mathrm{d}r}{R} = \frac{\rho_\mathrm{s}}{2\varepsilon_0}\int_{r=0}^{a}\mathrm{d}R = \frac{\rho_\mathrm{s}}{2\varepsilon_0}\,R\Big|_{r=0}^{a} = \frac{\rho_\mathrm{s}}{2\varepsilon_0}\left(\sqrt{a^2+z^2}-|z|\right)$$
$$(-\infty < z < \infty)\,. \tag{P1.77}$$

PROBLEM 1.35 **Potential due to a hemispherical surface charge.** We use the same subdivision of the hemispherical surface into thin rings as in Fig.1.16 and integrate the contributions to the potential by individual charged rings, as has been done for the electric field in Example 1.12. Substituting the expression for the charge of the ring shown in Fig.1.16, $\mathrm{d}Q = \rho_\mathrm{s}2\pi a^2 \sin\theta\,\mathrm{d}\theta$ [Eq.(1.65)], and $R = a$ in the expression for the potential due to the ring (previous problem), $\mathrm{d}V = \mathrm{d}Q/(4\pi\varepsilon_0 R)$, and integrating the such obtained expression as in Eq.(1.67), the resultant potential at the point O in Fig.1.16 turns out to be

$$V = \int_{\theta=0}^{\pi/2}\mathrm{d}V = \frac{\rho_\mathrm{s} a}{2\varepsilon_0}\int_0^{\pi/2}\sin\theta\,\mathrm{d}\theta = \frac{\rho_\mathrm{s} a}{2\varepsilon_0}\,. \tag{P1.78}$$

PROBLEM 1.36 **Potential due to a nonuniform spherical surface charge.** With respect to the potential computation in the previous problem, we now have a nonuniform charge density and different integration upper limit (full sphere), so that

$$V = \int_{\theta=0}^{\pi}\mathrm{d}V = \frac{a}{2\varepsilon_0}\int_0^{\pi}\rho_\mathrm{s}(\theta)\sin\theta\,\mathrm{d}\theta = \frac{\rho_{\mathrm{s}0} a}{2\varepsilon_0}\int_0^{\pi}\sin 2\theta\sin\theta\,\mathrm{d}\theta = 0\,. \tag{P1.79}$$

The potential at the center of the sphere is zero essentially because its total charge (Q) is zero.

Section 1.8 Voltage

PROBLEM 1.37 **Voltage due to two point charges.** Denoting one of the two vertices with no charge at them by A and the center of the square by B (Fig.P1.16) and combining Eqs.(1.88) and (1.80), the voltage between A and B comes out to be

$$V_\mathrm{AB} = V_\mathrm{A} - V_\mathrm{B} = \left(\frac{Q_1}{4\pi\varepsilon_0 a} + \frac{Q_2}{4\pi\varepsilon_0 a}\right) - \left(\frac{Q_1}{4\pi\varepsilon_0\sqrt{2}a/2} + \frac{Q_2}{4\pi\varepsilon_0\sqrt{2}a/2}\right)$$

$$= \frac{(1-\sqrt{2})(Q_1+Q_2)}{4\pi\varepsilon_0 a} = -99.27\ \text{kV}\ . \tag{P1.80}$$

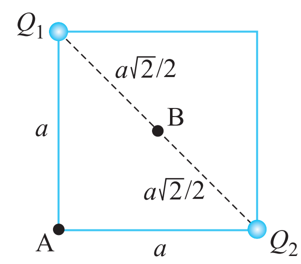

Figure P1.16 Finding the voltage between points A and B due to two point charges, Q_1 and Q_2, at nonadjacent vertices of a square.

Section 1.9 Differential Relationship Between the Field and Potential in Electrostatics

PROBLEM 1.38 **Sketch field from potential.** As in Eq.(1.99), the electric field intensity in the region is given by $E_x(x) = -\,\mathrm{d}V/\mathrm{d}x$, i.e., it equals the negative of the derivative of the function $V(x)$ at the coordinate x. In other words, $E_x(x)$ equals the negative of the slope of the $V(x)$ curve in Fig.1.54 at the corresponding abscissa point x, and based on this fact we sketch the function $E_x(x)$ – in Fig.P1.17.

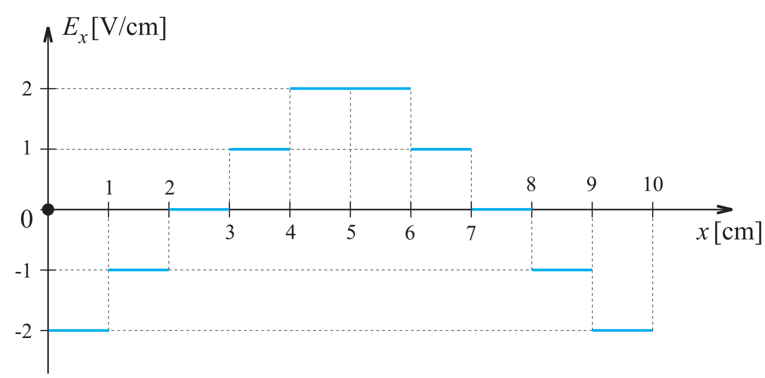

Figure P1.17 Plot of the electric field intensity $E_x(x)$ in the region with the 1-D potential distribution $V(x)$ given in Fig.1.54.

Section 1.10 Gradient

PROBLEM 1.39 **Field from potential, point charge.** We first represent the expression for the electric potential due to a point charge in free space, given by

Eq.(1.80), in the spherical coordinate system in which Q is at the coordinate origin, so simply replace R by r in Eq.(1.80). Then, we apply the formula for the gradient in spherical coordinates, Eq.(1.108), in conjunction with Eq.(1.101) to obtain the corresponding electric field vector, as follows (because V is a function of r only, the gradient formula retains only the first term):

$$\mathbf{E} = -\nabla V = -\frac{\mathrm{d}V}{\mathrm{d}r}\,\hat{\mathbf{r}} = -\frac{Q}{4\pi\varepsilon_0}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\right)\hat{\mathbf{r}} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{\mathbf{r}}\,. \qquad \text{(P1.81)}$$

This, of course, is the same result as in Eq.(1.24) with $R = r$ and $\hat{\mathbf{R}} = \hat{\mathbf{r}}$.

PROBLEM 1.40 **Field from potential, charged semicircle.** (a) A combination of Eqs.(1.101) and (1.102) and the expression for the electric potential from Problem 1.33, $V = Q'a/(4\varepsilon_0\sqrt{z^2+a^2})$, gives the following for the z-component of the electric field vector along the z-axis in Fig.1.12(a):

$$\mathbf{E} = -\nabla V = -\frac{\mathrm{d}V}{\mathrm{d}z}\,\hat{\mathbf{z}} = -\frac{Q'a}{4\varepsilon_0}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{\sqrt{z^2+a^2}}\right)\hat{\mathbf{z}} = \frac{Q'az}{4\varepsilon_0\left(z^2+a^2\right)^{3/2}}\,\hat{\mathbf{z}}\,, \qquad \text{(P1.82)}$$

which is the same result as in Eq.(1.50).

(b) Having in mind Eqs.(1.101) and (1.102), the x-component of the vector $\mathbf{E}$ due to the charged semicircle equals $E_x = -\,\mathrm{d}V/\mathrm{d}x$, and to compute this derivative at the point P in Fig.1.12(a), we need not only the value of the potential V at this point (or at points along the z-axis) but also at points off of the axis, in the x direction, which, however, is not provided by the expression for V from Problem 1.33.

PROBLEM 1.41 **Field from potential, charged disk.** In the same way as in Eq.(1.112), we combine Eqs.(1.101) and (1.105) and the expression for V given in Problem 1.34, $V = \rho_\mathrm{s}(\sqrt{a^2+z^2} - |z|)/(2\varepsilon_0)$, to obtain

$$\mathbf{E} = -\nabla V = -\frac{\mathrm{d}V}{\mathrm{d}z}\,\hat{\mathbf{z}} = -\frac{\rho_\mathrm{s}}{2\varepsilon_0}\frac{\mathrm{d}}{\mathrm{d}z}\left(\sqrt{a^2+z^2} - |z|\right)\hat{\mathbf{z}} = \frac{\rho_\mathrm{s}}{2\varepsilon_0}\left(\frac{z}{|z|} - \frac{z}{\sqrt{a^2+z^2}}\right)\hat{\mathbf{z}}\,, \qquad \text{(P1.83)}$$

namely, the same result as in Eq.(1.63), where the use is made of the fact that the derivative (slope) of the function $|z|$ equals 1 for $z > 0$, is -1 for $z < 0$, and is not defined for $z = 0$, which, exactly, is represented by the function $z/|z|$.

PROBLEM 1.42 **Field from potential, charged hemisphere.** It is impossible to find $\mathbf{E} = -\mathrm{grad}\,V$ at the hemisphere center (point O) in Fig.1.16 from the expression for the potential obtained in Problem 1.35 because this expression gives V only at that very point, which is not enough; to compute $\mathbf{E} = E_z\,\hat{\mathbf{z}} = -(\,\mathrm{d}V/\mathrm{d}z)\,\hat{\mathbf{z}}$ at the point O, we need the function $V(z)$ along the z-axis, at least in a small neighborhood of the point.

PROBLEM 1.43 **Angle between field lines and equipotential surfaces.** Since the potential does not change ($V = \text{const}$) over an equipotential surface, by its definition, the gradient of the potential, ∇V, and thus the electric field vector (in electrostatics), $\mathbf{E} = -\nabla V$ [Eq.(1.101)], does not have a component tangential to this surface (see Fig.1.27), i.e., $\mathbf{E}$ is perpendicular to it.

PROBLEM 1.44 **Direction of the steepest ascent.** (a) As the gradient of a scalar function at a location in space points in the direction in which the function increases the most at that location [Eq.(1.110) and Fig.1.27 for $\beta = 0$], the direction of the steepest ascent (maximum increase of the terrain elevation, h) at the location (x, y) is given by $\text{grad}\, h$, which results in [we first transform the expression for $h(x, y)$ such that x and y are entered in meters]

$$h(x,y) = 0.1x\ln(0.001y)\ \text{m} \quad (x,\ y \text{ in m}) \quad \longrightarrow \quad \nabla h(x,y) = \frac{\partial h}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial h}{\partial y}\,\hat{\mathbf{y}}$$

$$= 0.1\ln(0.001y)\,\hat{\mathbf{x}} + 0.1\,\frac{x}{y}\,\hat{\mathbf{y}} \quad (x,\ y \text{ in m})\,. \tag{P1.84}$$

At the specified location, $(3000\ \text{m}, 3000\ \text{m})$, the unit vector $\hat{\mathbf{l}}$ ($|\hat{\mathbf{l}}| = 1$) defining this direction (of the steepest ascent) turns out to be

$$x = y = 3000\ \text{m} \quad \longrightarrow \quad \nabla h = 0.1\,(\ln 3\,\hat{\mathbf{x}} + \hat{\mathbf{y}}) \quad \longrightarrow \quad \hat{\mathbf{l}} = \frac{\nabla h}{|\nabla h|}$$

$$= \frac{\ln 3\,\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{(\ln 3)^2 + 1}} = 0.7395\,\hat{\mathbf{x}} + 0.6731\,\hat{\mathbf{y}}\,. \tag{P1.85}$$

(b) From Eq.(1.111), the maximum space rate of increase in the function h per unit distance is equal to the magnitude of the vector ∇h at that location [$(\mathrm{d}h/\mathrm{d}l)_{\max} = |\nabla h|$]. Hence, using Eqs.(P1.85), the ascent at the point $(3000\ \text{m}, 3000\ \text{m})$, expressed as an angle α (in degrees), amounts to

$$\left.\frac{\Delta h}{\Delta l}\right|_{\max} = \left.\frac{\mathrm{d}h}{\mathrm{d}l}\right|_{\max} = |\nabla h| = 0.1\sqrt{(\ln 3)^2 + 1} = 0.1485$$

$$\longrightarrow \quad \alpha = \arctan\frac{\Delta h}{\Delta l} = \arctan 0.1485 = 8.45^\circ \tag{P1.86}$$

($\arctan \equiv \tan^{-1}$).

PROBLEM 1.45 **Maximum increase in electrostatic potential.** Using Eq.(1.110) and Fig.1.27 for $\beta = 0$, in conjunction with Eq.(1.101), the direction of the maximum increase in the electric potential at a point (x, y, z) is determined by $\text{grad}\, V$, as follows:

$$\text{Direction of } \left.\frac{\mathrm{d}V}{\mathrm{d}l}\right|_{\max} \quad \longrightarrow \quad \nabla V(x,y,z) = -\mathbf{E}(x,y,z)$$

$$= (-4\,\hat{\mathbf{x}} + z^2\,\hat{\mathbf{y}} - 2yz\,\hat{\mathbf{z}})\ \text{V/m} \quad (x,\ y,\ z \text{ in m})\,. \tag{P1.87}$$

At the given point, (1 m, 1 m, −1 m), the unit vector $\hat{\mathbf{l}}$ defining this direction is [see Eq.(1.5)]

$$x = y = -z = 1\ \text{m} \quad \longrightarrow \quad \nabla V = (-4\,\hat{\mathbf{x}} + \hat{\mathbf{y}} + 2\,\hat{\mathbf{z}})\ \text{V/m} \quad \longrightarrow \quad \hat{\mathbf{l}} = \frac{\nabla V}{|\nabla V|} =$$

$$= \frac{-4\,\hat{\mathbf{x}} + \hat{\mathbf{y}} + 2\,\hat{\mathbf{z}}}{\sqrt{(-4)^2 + 1^2 + 2^2}} = -0.8729\,\hat{\mathbf{x}} + 0.2182\,\hat{\mathbf{y}} + 0.4364\,\hat{\mathbf{z}}\,. \qquad \text{(P1.88)}$$

Section 1.11 3-D and 2-D Electric Dipoles

PROBLEM 1.46 Large and small electric dipole. (a) For the observation (potential/field) point midway between the two charges, application of Eqs.(1.80) and (1.24) and the superposition principle gives the following for the resultant electric potential and field vector due to the charges:

$$V = \frac{Q}{4\pi\varepsilon_0 R_1} + \frac{-Q}{4\pi\varepsilon_0 R_2} = 0\,, \quad \mathbf{E} = 2\,\frac{Q}{4\pi\varepsilon_0 R_1^2}\,(-\,\hat{\mathbf{z}}) = -\frac{Q}{2\pi\varepsilon_0 R_1^2}\,\hat{\mathbf{z}}$$

$$= -17.98\,\hat{\mathbf{z}}\ \text{V/m} \quad (Q_1 = -Q_2 = Q = 1\ \text{nC}\,,\ \ R_1 = R_2 = 1\ \text{m})\,. \qquad \text{(P1.89)}$$

(b) Next, the observation point and the two charges form a right-angled triangle, and hence [see a similar field computation in Eqs.(1.220) an Fig.1.48(b)]

$$V = \frac{1}{4\pi\varepsilon_0 R_1}(Q - Q) = 0\,, \quad \mathbf{E} = 2\,\frac{Q}{4\pi\varepsilon_0 R_1^2}\,\cos\alpha\,(-\,\hat{\mathbf{z}}) = -\frac{\sqrt{2}Q}{4\pi\varepsilon_0 R_1^2}\,\hat{\mathbf{z}}$$

$$= -6.355\,\hat{\mathbf{z}}\ \text{V/m} \quad \left(R_1 = R_2 = \sqrt{2}\ \text{m} = 1.414\ \text{m}\,,\ \ \alpha = 45^\circ\right). \qquad \text{(P1.90)}$$

(c) Finally, the distance of the observation point from the coordinate origin now being $r = \sqrt{100^2 + 100^2 + 100^2}$ m $= 100\sqrt{3}$ m $\gg d = 2$ m, the two point charges can be considered as a small electric dipole in Fig.1.28, of length $d = 2$ m. Therefore, we use Eqs.(1.115) and (1.117) for this dipole, and obtain

$$V = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2} = 346\ \mu\text{V}\,, \quad \mathbf{E} = \frac{Qd}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) = (4\,\hat{\mathbf{r}} + 2.82\,\hat{\boldsymbol{\theta}})\ \mu\text{V/m}$$

$$\left(d = 2\ \text{m}\,,\ \ r = 100\sqrt{3}\ \text{m} = 173.2\ \text{m}\,,\ \ \theta = \arccos\frac{1}{\sqrt{3}} = 54.74^\circ\right) \qquad \text{(P1.91)}$$

($\arccos \equiv \cos^{-1}$). Note, on the other hand, that the two-charge system in cases (a) and (b) is a large electric dipoles (the source-to-field distances are of a comparable size as the dipole), so that the dipole potential and field expressions derived in Section 1.11 (under the condition $r \gg d$) cannot be invoked.

PROBLEM 1.47 Potential and field due to a small electric dipole. (a)-(f) As $r \gg d$ in all cases, (a)-(f), the dipole potential and field are computed at the respective points (r, θ, ϕ) using Eqs.(1.115) and (1.117) as follows:

$$V_{\rm a} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = 9\ {\rm mV}\ ,\quad \mathbf{E}_{\rm a} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) = 18\,\hat{\mathbf{r}}\ {\rm mV/m}\quad (r = 1\ {\rm m}\,,$$
$$\theta = \phi = 0)\ ;\quad V_{\rm b} = 0\ ,\quad \mathbf{E}_{\rm b} = 9\,\hat{\boldsymbol{\theta}}\ {\rm mV/m}\quad (r = 1\ {\rm m}\,,\quad \theta = \phi = \pi/2)\ ;$$
$$V_{\rm c} = -9\ {\rm mV}\ ,\quad \mathbf{E}_{\rm c} = -18\,\hat{\mathbf{r}}\ {\rm mV/m}\ ;\quad V_{\rm d} = 6.35\ {\rm mV}\ ,$$
$$\mathbf{E}_{\rm d} = (12.7\,\hat{\mathbf{r}} + 6.35\,\hat{\boldsymbol{\theta}})\ {\rm mV/m}\ ;\quad V_{\rm e} = 63.5\ \mu{\rm V}\ ,\quad \mathbf{E}_{\rm e} = (12.7\,\hat{\mathbf{r}} + 6.35\,\hat{\boldsymbol{\theta}})\ \mu{\rm V/m}\ ;$$
$$V_{\rm f} = 0.635\ \mu{\rm V}\ ,\quad \mathbf{E}_{\rm f} = (12.7\,\hat{\mathbf{r}} + 6.35\,\hat{\boldsymbol{\theta}})\ {\rm nV/m}\ . \qquad \text{(P1.92)}$$

PROBLEM 1.48 Dipole equivalent to a nonuniform line charge. As shown in Problem 1.16, the total charge of the semicircle is zero, as is the case for an electric dipole, in Fig.1.28 [$Q_{\rm tot} = Q + (-Q) = 0$ for the dipole]. Since the charge density function of the semicircle, $Q'(\phi) = Q'_0 \sin\phi$, assuming that $Q'_0 > 0$, is positive for $0 < \phi \le \pi/2$ and negative for $-\pi/2 \le \phi < 0$, we expect that the positive point charge (Q) of the equivalent electric dipole replacing this nonuniform line charge should be on the positive part of the y-axis in Fig.1.12(a), while the negative point charge ($-Q$) is on the negative part of the axis.

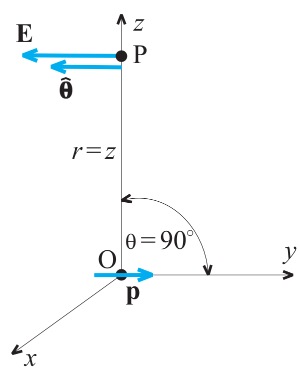

Figure P1.18 Electric dipole of moment **p** equivalent to the nonuniform line charge distribution along the semicircle [Fig.1.12(a)] from Problem 1.16 (equivalency at the field point P, where $|z| \gg a$).

If $|z| \gg a$ along the z-axis in Fig.1.12(a), $\sqrt{z^2 + a^2} \approx |z|$, with which the electric field vector due to the charged semicircle (Problem 1.16) becomes

$$\mathbf{E} = -\frac{Q'_0 a^2}{8\varepsilon_0 \left(z^2 + a^2\right)^{3/2}}\,\hat{\mathbf{y}} \approx -\frac{Q'_0 a^2}{8\varepsilon_0 |z|^3}\,\hat{\mathbf{y}}\qquad (|z| \gg a)\ . \qquad \text{(P1.93)}$$

For the spherical coordinate system whose z-axis is along the y-axis in Fig.1.12(a), **E** has only a θ-component, as depicted in Fig.P1.18. Comparing this field vector expression to that of an electric dipole in Eq.(1.117), we identify the moment (**p**) of the equivalent dipole to be

$$\mathbf{E} = \frac{Q'_0 a^2}{8\varepsilon_0 |z|^3}\,(-\,\hat{\mathbf{y}}) = \frac{Q'_0 a^2}{8\varepsilon_0 r^3}\,\sin 90^\circ\,\hat{\boldsymbol{\theta}} = \frac{p}{4\pi\varepsilon_0 r^3}\,\hat{\boldsymbol{\theta}}\quad\longrightarrow\quad \mathbf{p} = \frac{\pi Q'_0 a^2}{2}\,\hat{\mathbf{y}}\ . \qquad \text{(P1.94)}$$

PROBLEM 1.49 Expression for the electric field due to a line dipole. Similarly to the derivation in Eq.(1.117), we now use the formula for the gradient in cylindrical coordinates, Eq.(1.105), and apply it to the expression for V in Eq.(1.121), which gives the associated expression for $\mathbf{E}$ of a line dipole (Fig.1.29):

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\,\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial \phi}\,\hat{\boldsymbol{\phi}} = \frac{p'}{2\pi\varepsilon_0 r^2}\left(\cos\phi\,\hat{\mathbf{r}} + \sin\phi\,\hat{\boldsymbol{\phi}}\right) . \qquad \text{(P1.95)}$$

PROBLEM 1.50 Near and far potential and field due to a line dipole. This is a 2-D variation of the computation in Problem 1.46, and we refer to Fig.1.29, showing a cross section of a line dipole.

(a) As the distance of the observation point from the dipole axis, $r = 2$ m, is equal to the displacement between the dipole charges, $d = 2$ m, we cannot employ here line dipole potential and field expressions derived for the case $r \gg d$. Instead, we resort to Eqs.(1.119) and (1.57), and obtain

$$V = \frac{Q'}{2\pi\varepsilon_0}\ln\frac{r_2}{r_1} = \frac{Q'}{2\pi\varepsilon_0}\ln 3 = 1.975\ \text{V} , \quad \mathbf{E} = \frac{Q'}{2\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\hat{\mathbf{x}} = 1.198\,\hat{\mathbf{x}}\ \text{V/m}$$

$$(Q'_1 = -Q'_2 = Q' = 100\ \text{pC/m} ,\ \ r_1 = 1\ \text{m} ,\ \ r_2 = 3\ \text{m}) . \qquad \text{(P1.96)}$$

(b) As $r = 100\sqrt{2}$ m $\gg d = 2$ m, we now do invoke Eq.(1.121) and the associated expression for $\mathbf{E}$ (from the previous problem), $\mathbf{E} = Q'd(\cos\phi\,\hat{\mathbf{r}} + \sin\phi\,\hat{\boldsymbol{\phi}})/(2\pi\varepsilon_0 r^2)$, which yield

$$V = \frac{Q'd\cos\phi}{2\pi\varepsilon_0 r} = 17.98\ \text{mV} , \quad \mathbf{E} = \frac{Q'd}{2\pi\varepsilon_0 r^2}\left(\cos\phi\,\hat{\mathbf{r}} + \sin\phi\,\hat{\boldsymbol{\phi}}\right)$$

$$= 127.1(\hat{\mathbf{r}} + \hat{\boldsymbol{\phi}})\ \mu\text{V/m} \qquad \left(d = 2\ \text{m} ,\ \ r = 100\sqrt{2}\ \text{m} = 141.4\ \text{m} ,\ \ \phi = 45^\circ\right) . \qquad \text{(P1.97)}$$

Section 1.12 Formulation and Proof of Gauss' Law

PROBLEM 1.51 Flux of the electric field vector through a cube side. From Gauss' law, Eq.(1.133), the total outward flux of the electric field intensity vector due to the point charge Q through the (closed) surface S (all six sides) of the cube (Fig.P1.19) equals Q/ε_0 (the enclosed charge in S is $Q_S = Q$). Since the field lines due to charge Q are radial "beams" starting at the charge, Fig.1.22, and it is placed at the cube center, the outward fluxes of $\mathbf{E}$ through all sides of the cube are equal, and hence

$$(\Psi_E)_{\text{through one side}} = \frac{1}{6}\,(\Psi_E)_{\text{through all six sides}} = \frac{1}{6}\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{6}\,\frac{Q_S}{\varepsilon_0} = \frac{Q}{6\varepsilon_0} . \qquad \text{(P1.98)}$$

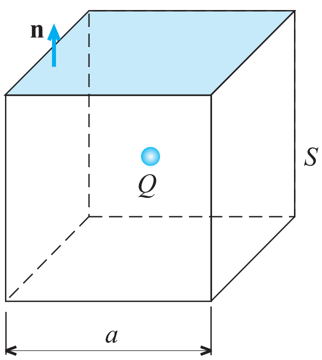

Figure P1.19 Point charge Q at the center of a cube in free space: evaluation of the outward flux of the electric field intensity vector through the top (or any other) side of the cube.

PROBLEM 1.52 Flux for a different placement of the point charge. Let us assume that the point charge Q is placed at the bottom side of the cube, which is horizontal, as shown in Fig.P1.20, and let us consider a plane containing that side, which divides the space into upper and lower half-spaces. It is then obvious that exactly a half of the electric-field lines emanating from the charge belong to the upper half-space, as well as that these lines pass through the remaining five sides of the cube, and thus constitute the total outward flux of the vector **E** through them, which gives

$$(\Psi_E)_{\text{through five sides}} = (\Psi_E)_{\text{in upper half}-\text{space}} = \frac{1}{2}(\Psi_E)_{\text{in entire space}} = \frac{Q}{2\varepsilon_0} . \tag{P1.99}$$

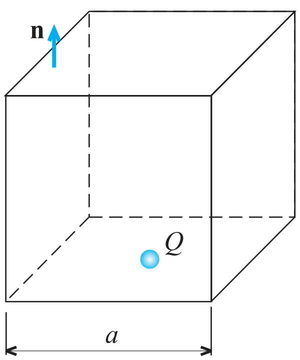

Figure P1.20 Point charge Q at the center of a side of the cube: evaluation of the total outward flux of **E** through the remaining five sides.

Section 1.13 Applications of Gauss' Law

PROBLEM 1.53 Field of a point charge from Gauss' law. This is a problem (the simplest one) with spherical symmetry. The vector **E** is of the form in

Eq.(1.136), the Gaussian surface S is a spherical surface in Fig.1.33, with the point charge Q at its center (point O), the outward flux of $\mathbf{E}$ through S is given in Eq.(1.138), the charge enclosed by S is simply $Q_S = Q$, and Gauss' law, Eq.(1.133), gives

$$\Psi_E = E(r)\,4\pi r^2 \quad \text{and} \quad \Psi_E = \frac{Q_S}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \quad \longrightarrow \quad \mathbf{E} = E(r)\,\hat{\mathbf{r}} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{\mathbf{r}} \,. \tag{P1.100}$$

PROBLEM 1.54 **Uniformly charged thin spherical shell.** (a) Because of spherical symmetry, the electric field vector everywhere (inside and outside the charged thin spherical shell) is given by $\mathbf{E} = E(r)\,\hat{\mathbf{r}}$ [Eq.(1.136)]. Applying Gauss' law to a spherical surface of radius r $(0 \le r < \infty)$ placed concentrically with the shell, we obtain the following equations for the two characteristic sizes of the surface:

$$E(r)\,4\pi r^2 = 0 \quad (0 \le r < a)\,, \quad E(r)\,4\pi r^2 = \frac{Q}{\varepsilon_0} \quad (a < r < \infty) \tag{P1.101}$$

(in the first case, there is no charge enclosed by the surface), and hence

$$\mathbf{E} = 0 \quad \text{(inside a spherical shell)} \quad \text{and} \quad \mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{\mathbf{r}} \quad \text{(outside)}\,. \tag{P1.102}$$

(b) As the field outside the shell is identical to that of a point charge Q, the same is true for the electric potential for $r > a$, including $V(a)$ at the very surface of the shell (for $r = a$). Hence, Eq.(1.80) directly results in the potential of the shell [also see Eq.(1.141)]:

$$V(a) = \frac{Q}{4\pi\varepsilon_0 a}\,. \tag{P1.103}$$

(c) Because the field is zero inside the shell, Eqs.(P1.102), the potential at the shell center (for $r = 0$) equals $V(a)$. Namely, by means of Eq.(1.90) or (1.142),

$$V(0) = \int_{r=0}^{a} E(r)\,\mathrm{d}r + V(a) = V(a) = \frac{Q}{4\pi\varepsilon_0 a}\,. \tag{P1.104}$$

PROBLEM 1.55 **Sphere with a nonuniform volume charge.** To find the distribution of the electric scalar potential inside and outside the nonuniformly charged sphere, we first compute the distribution of the electric field intensity vector everywhere – using Gauss' law. This problem is similar to both Examples 1.18 (sphere with a uniform charge) and 1.19 (cylinder with a nonuniform charge). Much like in Eq.(1.144), $\mathbf{E}$ inside the charge distribution is found applying Eq.(1.135) to a spherical Gaussian surface S of radius $r \le a$, shown in Fig.P1.21, as follows:

$$E_1(r)\,\underbrace{4\pi r^2}_{S} = \frac{1}{\varepsilon_0}\int_{r'=0}^{r} \underbrace{\rho_0\frac{r'}{a}}_{\rho}\,\underbrace{4\pi r'^2\,\mathrm{d}r'}_{\mathrm{d}v} \quad \longrightarrow \quad E_1(r) = \frac{\rho_0 r^2}{4\varepsilon_0 a}\,, \quad \text{for } r \le a\,, \tag{P1.105}$$

where $\mathrm{d}v$ is the volume of a thin spherical shell of radius r' ($0 < r' \leq r$) and thickness $\mathrm{d}r'$, given in Eq.(1.33). If the field point is outside the charged sphere ($r > a$), the upper limit in the integral in r' becomes a, resulting in

$$E_2(r) = \frac{\rho_0 a^3}{4\varepsilon_0 r^2} , \qquad \text{for } r > a . \tag{P1.106}$$

Invoking Eq.(1.74), the potential at a distance r of the sphere center (with respect to the reference point at infinity) is given by

$$V_2(r) = \int_{r'=r}^{\infty} E_2(r')\,\mathrm{d}r' = \frac{\rho_0 a^3}{4\varepsilon_0 r} , \qquad \text{for } r \geq a , \tag{P1.107}$$

and, similarly to the computation in Eq.(1.142),

$$V_1(r) = \int_{r'=r}^{a} E_1(r')\,\mathrm{d}r' + V_2(a) = \frac{\rho_0 a^2}{3\varepsilon_0}\left(1 - \frac{r^3}{4a^3}\right), \qquad \text{for } r < a . \tag{P1.108}$$

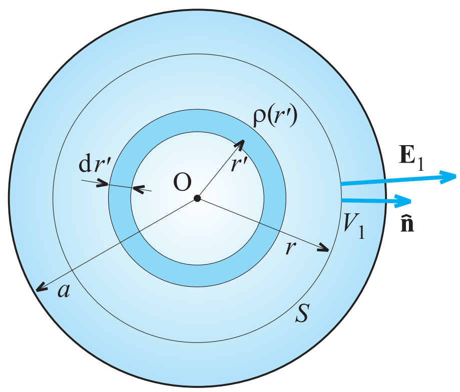

Figure P1.21 Evaluation of the electric field and potential due to a sphere with a nonuniform volume charge [ρ given in Eq.(1.32)].

PROBLEM 1.56 Field of an infinite line charge from Gauss' law. This being a problem (the simplest one) with cylindrical symmetry, the electric field is radial (with respect to the line charge), and Gauss' law [Eq.(1.133)] applied to the cylindrical surface of radius r ($a < r < \infty$) and height h, coaxial with the charge [see the left-hand side of Eq.(1.144)], yields

$$E(r)\underbrace{2\pi r h}_{S_\mathrm{c}} = \frac{1}{\varepsilon_0}\underbrace{Q'h}_{Q_S} \quad \longrightarrow \quad \mathbf{E} = E(r)\,\hat{\mathbf{r}} = \frac{Q'}{2\pi\varepsilon_0 r}\,\hat{\mathbf{r}} , \tag{P1.109}$$

where $\hat{\mathbf{r}}$ is the radial cylindrical unit vector.

PROBLEM 1.57 Uniformly charged thin cylindrical shell. This is the cylindrical version of Problem 1.54 (charged thin spherical shell). The electric field is radial with respect to the axis of the charged cylindrical shell, and we apply Gauss' law [Eq.(1.133)] to the cylindrical surface of radius r ($0 \le r < \infty$) and height h, coaxial with the shell. For the field point inside the shell,

$$E(r)\,2\pi r h = 0 \quad \longrightarrow \quad \mathbf{E} = 0 \quad (0 \le r < a) \tag{P1.110}$$

(there is no charge enclosed by the Gaussian surface). The total charge of the part of the shell of height (length) h being $Q_S = \rho_s 2\pi a h$ (surface charge density times the lateral surface area of this part of the cylinder of radius a), the field vector outside the shell turns out to be

$$E(r)\,2\pi r h = \frac{\rho_s 2\pi a h}{\varepsilon_0} \quad \longrightarrow \quad \mathbf{E} = E(r)\,\hat{\mathbf{r}} = \frac{\rho_s a}{\varepsilon_0 r}\,\hat{\mathbf{r}} \quad (a < r < \infty)\,. \tag{P1.111}$$

PROBLEM 1.58 **Cylinder with uniform volume charge.** Having in mind Eq.(1.144), the electric field intensity inside the uniformly charged infinite cylinder is given by

$$E(r)\,2\pi r h = \frac{1}{\varepsilon_0}\,\rho \underbrace{\pi r^2 h}_{v} \quad \longrightarrow \quad E(r) = \frac{\rho r}{2\varepsilon_0} \quad (0 \le r \le a) \tag{P1.112}$$

(since $\rho = \text{const}$, the charge enclosed by the Gaussian surface equals ρ times the volume v of the enclosed cylindrical domain). Employing then Eq.(1.90), the voltage between the surface and the axis of the cylinder, computed as the negative of the voltage between the axis and the surface, amounts to

$$V = -\int_{r=0}^{a} E(r)\,\mathrm{d}r = -\frac{\rho a^2}{4\varepsilon_0}\,. \tag{P1.113}$$

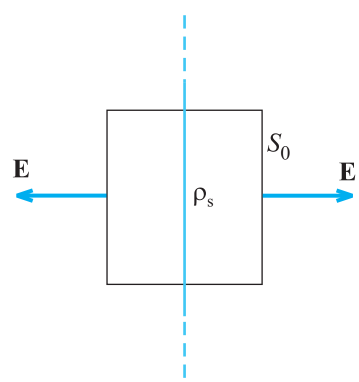

Figure P1.22 Application of Gauss' law to compute the electric field of an infinite sheet of charge.

PROBLEM 1.59 **Field of an infinite sheet of charge from Gauss' law.** This is a problem (the simplest one) with planar symmetry, so the electric field intensity vector everywhere is of the form in Eq.(1.148). We apply Gauss' law, Eq.(1.133), to a rectangular box shown in Fig.P1.22 (see also Fig.1.15), in a similar fashion as in Fig.1.35, and obtain [see the left-hand side of Eq.(1.149)]

$$2ES_0 = \frac{1}{\varepsilon_0} \underbrace{\rho_s S_0}_{Q_S} \longrightarrow E = \frac{\rho_s}{2\varepsilon_0} . \quad \text{(P1.114)}$$

PROBLEM 1.60 **Two parallel oppositely charged sheets.** (a) This is the simplest case of planar symmetry with an antisymmetrical charge distribution, like the one analyzed in Example 1.21. The total charge of the two sheets is zero, and the electric field outside the region between the sheets is zero as well. For the Gaussian rectangular surface portrayed in Fig.P1.23, what we obtain is the left-hand side of Gauss' law in Eq.(1.155) and the right-hand side as in the previous problem, so the electric field intensity E between the sheets turns out to be

$$ES_0 = \frac{1}{\varepsilon_0} \rho_s S_0 \longrightarrow E = \frac{\rho_s}{\varepsilon_0} . \quad \text{(P1.115)}$$

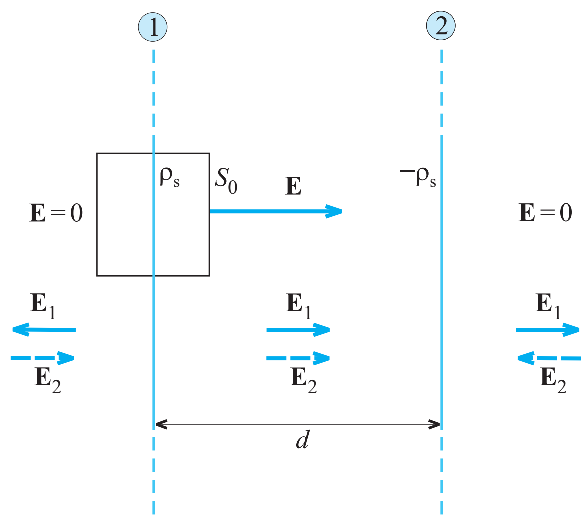

Figure P1.23 Evaluation of the electric field due to two parallel infinite sheets of opposite charges.

Alternatively, we can obtain the same result by the superposition of fields $E = \rho_s/(2\varepsilon_0)$ [Eq.(1.64)] due to the positively and negatively charged infinite sheets considered independently, which is also illustrated in Fig.P1.23, where

$$E = E_1 + E_2 = 2E_1 = 2\,\frac{\rho_s}{2\varepsilon_0} = \frac{\rho_s}{\varepsilon_0} \quad \text{(between sheets)} ,$$

$$E = E_1 - E_2 = 0 \quad \text{(outside)} . \quad \text{(P1.116)}$$

(b) The voltage between the sheets equals

$$V = Ed = \frac{\rho_s d}{\varepsilon_0} , \quad \text{(P1.117)}$$

with d being the distance between the sheets in Fig.P1.23.

PROBLEM 1.61 **Equivalent sheet of charge.** (a) As this is a problem with planar symmetry and symmetrical charge distribution, we refer to Fig.1.35 and Eq.(1.149), and write, for the field point inside the charge layer,

$$2\,E_x(x)\,S_0 = \frac{1}{\varepsilon_0}\,\rho\,\underbrace{S_0 2x}_{v} \quad\longrightarrow\quad E_x(x) = \frac{\rho x}{\varepsilon_0} \quad \left(-\frac{d}{2} \leq x \leq \frac{d}{2}\right), \tag{P1.118}$$

where v is the volume of the Gaussian rectangular box in Fig.1.35 and $Q_S = \rho v$ since $\rho = \text{const}$.

(b) For the field point outside the layer, as in Eqs.(1.151) and (1.152),

$$E_x(x) = E_x(d/2) = \frac{\rho d}{2\varepsilon_0} \quad \left(x > \frac{d}{2}\right), \quad E_x(x) = E_x(-d/2) = -\frac{\rho d}{2\varepsilon_0} \quad \left(x < -\frac{d}{2}\right). \tag{P1.119}$$

Comparing this result to that in Eq.(1.64), we realize that the surface charge density of the equivalent infinite sheet of charge (Fig.1.15) that can replace the layer, as far as the field outside it is concerned, amounts to $\rho_s = \rho d$.

PROBLEM 1.62 **Layer with a cosine volume charge distribution.** (a) The charge density $\rho(x)$ is an even function of the coordinate x (symmetrical charge distribution), and Eq.(1.149) becomes

$$2\,E_x(x)\,S_0 = \frac{1}{\varepsilon_0}\int_{x'=-x}^{x} \rho_0 \cos\left(\frac{\pi}{a}\,x'\right) S_0\,\mathrm{d}x' \quad\longrightarrow\quad E_x(x) = \frac{\rho_0 a}{\pi\varepsilon_0}\,\sin\left(\frac{\pi}{a}\,x\right)$$
$$(|x| \leq a)\ . \tag{P1.120}$$

Eqs.(1.151) and (1.152) then give

$$E_x(x) = E_x(a) = E_x(-a) = 0 \qquad (|x| > a)\ . \tag{P1.121}$$

(b) Since $E_x(x)$ is an odd function of x, the voltage across the charge layer (between planes $x = -a$ and $x = a$) comes out to be zero,

$$V = \int_{x=-a}^{a} E_x(x)\,\mathrm{d}x = \int_{-a}^{a} \sin\left(\frac{\pi}{a}\,x\right) \mathrm{d}x = 0\ . \tag{P1.122}$$

PROBLEM 1.63 **Layer with a sine charge distribution.** (a) The function $\rho(x)$ is now odd (antisymmetrical charge distribution), the electric field outside the charge layer is zero, Eq.(1.154), and that inside the layer is given by Eq.(1.155), which results in

$$E_x(x) = \frac{1}{\varepsilon_0}\int_{x'=-a}^{x} \rho_0 \sin\left(\frac{\pi}{a}\,x'\right) \mathrm{d}x' = -\frac{\rho_0 a}{\pi\varepsilon_0}\left[1 + \cos\left(\frac{\pi}{a}\,x\right)\right] \quad (|x| \leq a)\ . \tag{P1.123}$$

(b) The voltage between planes $x = -a$ and $x = a$ amounts to

$$V = \int_{x=-a}^{a} E_x(x)\,\mathrm{d}x = -\frac{2\rho_0 a^2}{\pi\varepsilon_0} \,. \tag{P1.124}$$

PROBLEM 1.64 Exponential charge distribution in the entire space. This is the same type of problem (planar with antisymmetrical charge distribution) as the previous one. However, the principal differences important for the solution procedure in the present case are the infinite extent of the charge distribution (it now starts and ends at infinity, along the x-axis), as well as its discontinuity at $x' = 0$. Hence, we position the left face of the Gaussian surface (in Fig.1.35) and start the volume integration in the plane $x' \to -\infty$ (where the charge distribution begins). In addition, because of the discontinuity in the function $\rho(x')$, we must separately analyze the cases when the field point is in the left half-space ($-\infty < x \le 0$) and in the right-hand one ($0 < x < \infty$). Finally, in the latter case, we must break the integration at the discontinuity point (plane), $x' = 0$. Having all these facts in mind, the algebraic intensity of the field vector $\mathbf{E}$ at a location defined by a negative (or zero) coordinate x is given by

$$E_x(x) = \frac{1}{\varepsilon_0}\int_{x'=-\infty}^{x} \rho_0\,\mathrm{e}^{x'/a}\,\mathrm{d}x' = \frac{\rho_0 a}{\varepsilon_0}\,\mathrm{e}^{x/a} \quad (x \le 0) \,, \tag{P1.125}$$

whereas for the positive x,

$$E_x(x) = \frac{1}{\varepsilon_0}\left[\int_{-\infty}^{0} \rho_0\,\mathrm{e}^{x'/a}\,\mathrm{d}x' + \int_{0}^{x}(-\rho_0)\,\mathrm{e}^{-x'/a}\,\mathrm{d}x'\right] = \frac{\rho_0 a}{\varepsilon_0}\,\mathrm{e}^{-x/a} \quad (x > 0) \,. \tag{P1.126}$$

Both results can be unified into a single field expression for the entire space:

$$\mathbf{E} = \frac{\rho_0 a}{\varepsilon_0}\,\mathrm{e}^{-|x|/a}\,\hat{\mathbf{x}} \quad (-\infty < x < \infty) \,. \tag{P1.127}$$

Section 1.14 Differential Form of Gauss' Law

PROBLEM 1.65 Uniform electric field. Since the electric field in the region is uniform ($\mathbf{E}_0 = \text{const}$), all spatial derivatives of $\mathbf{E}_0$ are zero, and Gauss' law in differential form, Eq.(1.163), tells us that there is no excess volume charge in the region, namely, $\rho = 0$.

PROBLEM 1.66 Charge distribution from 1-D field distribution. As the field vector $\mathbf{E}$ in this electrostatic system has an x-component only, and E_x depends on the coordinate x only (one-dimensional field distribution), we use the 1-D Gauss' law in differential form, Eq.(1.158). Combined with Eq.(1.99), it gives the following charge density in the system:

$$\rho = \varepsilon_0 \frac{\mathrm{d}E_x}{\mathrm{d}x} = -\frac{2\varepsilon_0 V_2}{d^2} \ . \qquad \text{(P1.128)}$$

Section 1.15 Divergence

PROBLEM 1.67 **Charge from field, planar symmetry.** We apply the one-dimensional differential Gauss' law – in Eq.(1.158). From Eq.(1.150),

$$\rho = \varepsilon_0 \frac{\mathrm{d}E_x}{\mathrm{d}x} = \rho_0 \left(1 - \frac{x^2}{a^2}\right) , \qquad \text{for } |x| \leq a \ , \qquad \text{(P1.129)}$$

which, of course, is the charge density in Eq.(1.147). From Eqs.(1.151) and (1.152), on the other side,

$$\rho = \varepsilon_0 \frac{\mathrm{d}E_x}{\mathrm{d}x} = 0 \ , \qquad \text{for } |x| > a \ . \qquad \text{(P1.130)}$$

PROBLEM 1.68 **Charge from field, cylindrical symmetry.** A combination of the differential Gauss' law in Eq.(1.166) and the formula for the divergence in cylindrical coordinates, Eq.(1.170), applied to the field expressions in Eqs.(1.145) and (1.146), gives (because $\mathbf{E}$ has a radial cylindrical component only, $E = E_r$, the divergence formula retains only the first term):

$$\rho = \varepsilon_0 \nabla \cdot \mathbf{E} = \frac{\varepsilon_0}{r} \frac{\partial}{\partial r}(rE_r) = \frac{1}{r} \frac{\partial}{\partial r}\left(\frac{\rho_0 r^4}{4a^2}\right) = \frac{\rho_0 r^2}{a^2} \ , \qquad \text{for } r \leq a \ ,$$

$$\rho = \frac{\varepsilon_0}{r} \frac{\partial}{\partial r}(rE_r) = \frac{1}{r} \frac{\partial}{\partial r}\left(\frac{\rho_0 a^2}{4}\right) = 0 \ , \qquad \text{for } r > a \ , \qquad \text{(P1.131)}$$

that is, the charge distribution in Eq.(1.143).

PROBLEM 1.69 **Charge from field, spherical symmetry.** This is a version of the previous problem but in spherical coordinates. Eqs.(1.166), (1.171), and (1.140) lead to

$$\rho = \varepsilon_0 \nabla \cdot \mathbf{E} = \frac{\varepsilon_0}{r^2} \frac{\partial}{\partial r}(r^2 E_r) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(\frac{\rho r^3}{3}\right) = \rho = \text{const} \ , \qquad \text{for } r \leq a \ ,$$

$$\rho = \frac{\varepsilon_0}{r} \frac{\partial}{\partial r^2}(r^2 E_r) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(\frac{\rho a^3}{3}\right) = 0 \ , \qquad \text{for } r > a \ , \qquad \text{(P1.132)}$$

which indeed represents a uniformly charged sphere of radius a and charge density ρ in free space.

PROBLEM 1.70 Nonuniformly charged sphere using differential Gauss' law. This is a version of Example 1.22 but with a nonuniform charge distribution in a sphere, so a function $\rho(r)$, Eq.(1.32), in place of a constant ρ. For the field point inside the charge distribution, Eq.(1.174) now takes form

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 E(r)\right] = \frac{\rho(r)}{\varepsilon_0} = \frac{\rho_0 r}{\varepsilon_0 a} \quad (0 \le r \le a) \ . \tag{P1.133}$$

We solve this differential equation in r by integration, as in Eqs.(1.175),

$$r^2 E(r) = \frac{\rho_0}{\varepsilon_0 a}\int r^3 \, \mathrm{d}r = \frac{\rho_0 r^4}{4\varepsilon_0 a} + C_1 \quad \longrightarrow \quad E(r) = \frac{\rho_0 r^2}{4\varepsilon_0 a} + \frac{C_1}{r^2} = \frac{\rho_0 r^2}{4\varepsilon_0 a} \ , \tag{P1.134}$$

with the integration constant C_1 being zero – from the "initial" condition $E(0) = 0$.

For the field point outside the charged sphere, where $\rho(r) = 0$, the differential equation and its solution are exactly those in Eqs.(1.176), whereas the integration constant C_2 is determined from the "boundary" condition at the sphere boundary, as follows:

$$E(r) = \frac{C_2}{r^2} \quad \text{and} \quad E(a^-) = E(a^+) = E(a) \quad \longrightarrow \quad C_2 = \frac{\rho_0 a^3}{4\varepsilon_0}$$

$$\longrightarrow \quad E(r) = \frac{\rho_0 a^3}{4\varepsilon_0 r^2} \quad (a < r < \infty) \ . \tag{P1.135}$$

We note that the results in Eqs.(P1.134) and (P1.135) are, of course, the same as those obtained using Gauss' law in integral form in Problem 1.55.

PROBLEM 1.71 Problem with cylindrical symmetry by differential Gauss' law. We now use the formula for the divergence in cylindrical coordinates, Eq.(1.170), which incorporated in Gauss' law in differential form, Eq.(1.166), gives the following differential equation in the radial cylindrical coordinate r for the field point inside the charge distribution in Fig.1.34:

$$\nabla \cdot \mathbf{E} = \frac{1}{r}\frac{\partial}{\partial r}\left[r E(r)\right] = \frac{\rho(r)}{\varepsilon_0} = \frac{\rho_0 r^2}{\varepsilon_0 a^2} \quad (0 \le r < a) \ . \tag{P1.136}$$

By its integration, we obtain

$$r E(r) = \frac{\rho_0}{\varepsilon_0 a^2}\int r^3 \, \mathrm{d}r = \frac{\rho_0 r^4}{4\varepsilon_0 a^2} + C_1 \quad \longrightarrow \quad E(r) = \frac{\rho_0 r^3}{4\varepsilon_0 a^2} + \frac{C_1}{r} \ , \tag{P1.137}$$

where $C_1 = 0$, because $E(0) = 0$ [note that if there were a line charge of density Q_0' along the axis of the cylinder in Fig.1.34, this constant would amount to $C_1 = Q_0'/(2\pi\varepsilon_0)$, from Eq.(1.57)].

If the field point is outside the charged cylinder in Fig.1.34 ($\rho = 0$), the differential equation and its solution become

$$\frac{\partial}{\partial r}\left[r E(r)\right] = 0 \quad \longrightarrow \quad r E(r) = C_2 \quad \longrightarrow \quad E(r) = \frac{C_2}{r} \quad (a < r < \infty) \ , \tag{P1.138}$$

and the constant C_2 amounts to

$$E(a^-) = E(a^+) = E(a) \quad \longrightarrow \quad C_2 = \frac{\rho a^2}{4\varepsilon_0} \ . \tag{P1.139}$$

The results for $E(r)$ in both regions, with the computed values of C_1 and C_2 substituted in Eqs.(P1.137) and (P1.138), are in agreement with Eqs.(1.145) and (1.146).

PROBLEM 1.72 Problem with planar symmetry using differential Gauss' law. This is a version of Example 1.23 but with a uniform charge distribution in a layer between $x = -a = -d/2$ and $x = a = d/2$ in Fig.1.35.

(a) For the field point inside the charge layer, Eq.(1.178) now becomes

$$\frac{\mathrm{d}E_x}{\mathrm{d}x} = \frac{\rho}{\varepsilon_0} \quad (\rho = \text{const}) \quad \longrightarrow \quad E_x(x) = \frac{1}{\varepsilon_0}\int \rho\,\mathrm{d}x = \frac{\rho x}{\varepsilon_0} + C_1 \quad \left(|x| \le \frac{d}{2}\right). \tag{P1.140}$$

Because of symmetry of the charge distribution with respect to the plane $x = 0$ in Fig.1.35,

$$E_x(a) = -E_x(-a) \quad \left(a = \frac{d}{2}\right) \quad \longrightarrow \quad \frac{\rho d}{2\varepsilon_0} + C_1 = \frac{\rho d}{2\varepsilon_0} - C_1 \quad \longrightarrow \quad C_1 = 0\,, \tag{P1.141}$$

with which the result in Eq.(P1.140) matches that obtained in Problem 1.61, part (a).

(b) If the observation point falls outside the charge layer ($|x| > d/2$), $\rho = 0$, and we have

$$\frac{\mathrm{d}E_x}{\mathrm{d}x} = 0 \quad \longrightarrow \quad E_x(x) = C_2 \quad \left(|x| > \frac{d}{2}\right). \tag{P1.142}$$

From the the "boundary" condition at the layer boundary defined by $x = a = d/2$,

$$E_x(a^+) = E_x(a^-) = E_x(a) \quad \longrightarrow \quad C_2 = \frac{\rho d}{2\varepsilon_0} \quad \longrightarrow \quad E_x(x) = \frac{\rho d}{2\varepsilon_0} \quad \left(x > \frac{d}{2}\right). \tag{P1.143}$$

On the other side of the layer, the vector $\mathbf{E}$ has this same magnitude but opposite direction, resulting in $E_x(x) = -\rho d/(2\varepsilon_0)$ $(x < -d/2)$. These solutions also are the same as in Problem 1.61, part (b).

PROBLEM 1.73 Antisymmetrical charge, differential Gauss' law. As shown in Eq.(1.154) and explained via a superposition of fields contributed by differentially thin layers constituting the charge distribution, the electric field outside a (thick) layer with an antisymmetrical charge distribution [$\rho(x)$ an odd function of x in Fig.1.35] must be zero. For the field point inside the layer with $\rho(x)$ given by Eq.(1.153),

$$\frac{\mathrm{d}E_x(x)}{\mathrm{d}x} = \frac{\rho(x)}{\varepsilon_0} = \frac{\rho_0 x}{\varepsilon_0 a} \quad \longrightarrow \quad E_x(x) = \frac{\rho_0}{\varepsilon_0 a}\int x\,\mathrm{d}x = \frac{\rho_0 x^2}{2\varepsilon_0 a} + C \quad (|x| \le a)\,. \tag{P1.144}$$

From the fact that the field outside is zero and the "boundary" condition at the layer boundary defined by $x = a$ (or $x = -a$),

$$E_x(a^+) = E_x(a^-) = E_x(a) = 0 \quad \longrightarrow \quad \frac{\rho_0 a}{2\varepsilon_0} + C = 0 \quad \longrightarrow \quad C = -\frac{\rho_0 a}{2\varepsilon_0}\,, \tag{P1.145}$$

which, substituted in Eq.(P1.144), gives the result for $E_x(x)$ in Eq.(1.155).

PROBLEM 1.74 **Gauss' law in differential and integral form.** (a) Using the differential Gauss' law in Eq.(1.163) or (1.166), the volume charge density in the region is given by

$$\rho = \varepsilon_0 \nabla \cdot \mathbf{E} = \varepsilon_0 \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = 4\varepsilon_0 y \ (\mathrm{C/m^3}) \quad (y \text{ in m}) \, . \qquad \text{(P1.146)}$$

(b) Performing volume integration with $\mathrm{d}v$ in the form of a slice of the cube with thickness $\mathrm{d}y$, the total charge enclosed in the cube amounts to

$$Q_S = \int_v \rho(y) \, \mathrm{d}v = \int_{y=0}^{1 \text{ m}} 4\varepsilon_0 y \underbrace{1^2 \, \mathrm{d}y}_{\mathrm{d}v} = 2\varepsilon_0 \ (\mathrm{C}) \, . \qquad \text{(P1.147)}$$

(c) The outward flux of the vector $\mathbf{E}$ (its y-component) through the cube side defined by $y = 0$ and that through the side $y = 1$ m cancel each other, and the same is true for the fluxes through the pair of sides perpendicular to the z-axis. The flux through the side $x = 0$ is zero because $E_x = 0$ on that side. Therefore, the net outward flux of $\mathbf{E}$ through the entire (closed) surface of the cube reduces to the outward flux (in the positive x direction) of $E_x \, \hat{\mathbf{x}}$ through the cube side at $x = 1$ m, which equals

$$\Psi_E = \oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \int_{\text{side } x=1 \text{ m}} E_x|_{x=1 \text{ m}} \, \hat{\mathbf{x}} \cdot \mathrm{d}S \, \hat{\mathbf{x}} = \int_{y=0}^{1 \text{ m}} 4y \underbrace{1 \, \mathrm{d}y}_{\mathrm{d}S} = 2 \ \mathrm{V \cdot m} \, , \quad \text{(P1.148)}$$

where $\mathrm{d}S$ is adopted to be a strip of width $\mathrm{d}y$ on this side. We see that, indeed, the computed Ψ_E [in Eq.(P1.148)] and Q_S [in Eq.(P1.147)] satisfy the relationship $\Psi_E = Q_S/\varepsilon_0$, which confirms the validity of both Gauss' law in integral form, Eq.(1.135), and the divergence theorem, Eq.(1.173), for this particular charge distribution and closed surface.

Section 1.17 Evaluation of the Electric Field and Potential Due to Charged Conductors

PROBLEM 1.75 **Excentric charged sphere inside an uncharged shell.** (a) The charged metallic sphere in Fig.1.41 is now excentric relative to the uncharged metallic spherical shell, but they are not touching. The total induced charge on the outer surface of the shell, which is a result of the electrostatic induction, amounts to $Q_c = -Q_b = Q_a = Q$ [from Eqs.(1.199) and (1.198)], as in the original structure (with concentric sphere), and this charge is uniformly distributed over the surface [as in Fig.1.41], because it is smooth and symmetrical (spherical). Therefore, the electric-field lines outside the shell, for $c < r < \infty$, remain as in Fig.1.41 [$E(r)$ is given in Eq.(1.200)], and the same is true for the electric potential of the shell.

Having in mind Eq.(1.202), this potential comes out to be

$$V_{\text{shell}} = \int_{r=c}^{\infty} E(r)\,\mathrm{d}r = \int_{c}^{\infty} \frac{Q_c}{4\pi\varepsilon_0 r^2}\,\mathrm{d}r = \frac{Q}{4\pi\varepsilon_0 c} \quad (Q_c = Q) \ . \tag{P1.149}$$

Note, however, that charges on the other two surfaces, i.e., on the surface of the sphere and the inner surface of the shell, although retaining their total values, $Q_a = Q$ and $Q_b = -Q_a = -Q$, respectively, as in the original structure, are not distributed uniformly any more in the new electrostatic state, and the electric field in the region between the sphere and the shell is not given by Eq.(1.200).

(b) If the sphere is pressed against the shell wall, the charge Q of the sphere flows to the periphery of the shell, so that $Q_c = Q$ (however, $Q_a = Q_b = 0$ on the other two surfaces). This charge is again uniformly distributed over the outer surface of the shell, and the field outside the shell remains the same. The potential of the shell is thus that in Eq.(P1.149).

PROBLEM 1.76 **Point charge inside a charged shell.** (a) Similarly to the analysis in Example 1.27 [Eqs.(1.198) and (1.199)], the total induced charges on the inner and outer surfaces of the shell are $Q_a = -2Q$ and $Q_b = Q - Q_a = 3Q$, respectively, and the charge distributions over the surfaces are illustrated in Fig.P1.24.

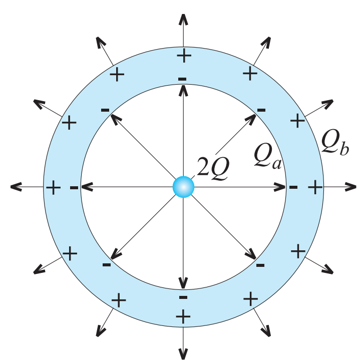

Figure P1.24 Surface-charge and field distributions in a system with a point charge inside a charged spherical metallic shell.

(b) Inspecting the field lines or using Gauss' law, as in Eq.(1.200), we obtain the following expression for the electric field intensity outside the shell in Fig.P1.24:

$$E(r) = \frac{Q_b}{4\pi\varepsilon_0 r^2} = \frac{3Q}{4\pi\varepsilon_0 r^2} \quad (b < r < \infty) \ , \tag{P1.150}$$

and hence the potential of the shell

$$V_{\text{shell}} = \int_{r=b}^{\infty} E(r)\,\mathrm{d}r = \frac{3Q}{4\pi\varepsilon_0} \int_{b}^{\infty} \frac{\mathrm{d}r}{r^2} = \frac{3Q}{4\pi\varepsilon_0 b} \ . \tag{P1.151}$$

PROBLEM 1.77 Three concentric shells, one uncharged. (a) Denoting the unknown charge of the middle spherical metallic shell by $Q_{\rm m}$ and having in mind that the charge of the outer shell is $Q_{\rm o} = 0$ (it is uncharged), the electric field intensities in the three characteristic air-filled regions in the system are given by

$$E_1(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \quad (a < r < b) \ , \quad E_2(r) = \frac{Q + Q_{\rm m}}{4\pi\varepsilon_0 r^2} \quad (c < r < d) \ ,$$
$$E_3(r) = \frac{Q + Q_{\rm m}}{4\pi\varepsilon_0 r^2} \quad (e < r < \infty) \ . \tag{P1.152}$$

The given potential of the middle shell with respect to the reference point at infinity can now be expressed as

$$V = \int_{r=c}^{d} E_2(r)\,{\rm d}r + \int_{e}^{\infty} E_3(r)\,{\rm d}r = \frac{(Q + Q_{\rm m})(cd - ce + de)}{4\pi\varepsilon_0 cde} = 1\ {\rm kV} \ , \tag{P1.153}$$

and solving this for $Q_{\rm m}$, we obtain $Q_{\rm m} = -2.85$ nC.

(b) The voltage between the inner and the outer shells is then

$$V_{\rm io} = \int_{a}^{b} E_1(r)\,{\rm d}r + \int_{c}^{d} E_2(r)\,{\rm d}r = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q(b-a)}{ab} + \frac{(Q + Q_{\rm m})(d-c)}{cd}\right] = 1.55\ {\rm kV} \, . \tag{P1.154}$$

PROBLEM 1.78 Three concentric shells, two at the same potential. (a) We note a different notation (for the same structure) in this problem with respect to the previous one. As the inner and outer shells are now at the same potential ($V_1 = V_3$), the voltage between them is zero, so that (previous problem)

$$0 = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1(b-a)}{ab} + \frac{(Q_1 + Q_2)(d-c)}{cd}\right] , \tag{P1.155}$$

from which the charge of the middle shell is found to be $Q_2 = -6.8$ nC.

(b) The potential of the inner shell with respect to the reference point at infinity (V_1) can be computed as that of the outer shell (V_3), because they are the same, as follows:

$$V_1 = V_3 = \int_{e}^{\infty} E_3(r)\,{\rm d}r = \frac{Q_1 + Q_2 + Q_3}{4\pi\varepsilon_0 e} = -611\ {\rm V} \ . \tag{P1.156}$$

Finally, using V_3, the potential of the middle shell amounts to

$$V_2 = \int_{c}^{d} E_2(r)\,{\rm d}r + V_3 = \frac{(Q_1 + Q_2)(d-c)}{4\pi\varepsilon_0 cd} + V_3 = -851\ {\rm V} \ . \tag{P1.157}$$

PROBLEM 1.79 Four coaxial cylindrical conductors. This is, essentially, a cylindrical-geometry version of Example 1.28. By virtue of Gauss' law [also see Eq.(1.196)], the electric field intensities in the three characteristic air-filled regions in the system, with the notation shown in Fig.P1.25, are given by

$$E_1(r) = \frac{Q_1'}{2\pi\varepsilon_0 r} \quad (2d < r < 3d) \ , \quad E_2(r) = \frac{Q_1'}{2\pi\varepsilon_0 r} \quad (4d < r < 5d) \ ,$$

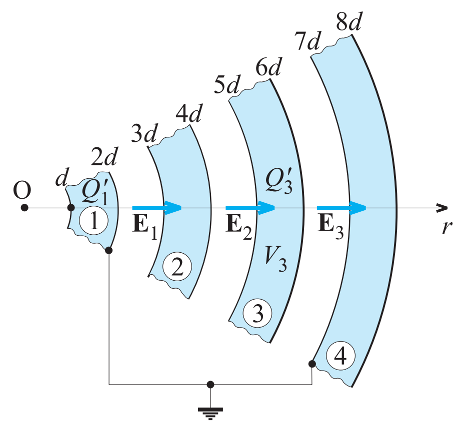

Figure P1.25 Electric field vectors in the system of four cylindrical conductors in Fig.1.55 (cross section of the system).

$$E_3(r) = \frac{Q_1' + Q_3'}{2\pi\varepsilon_0 r} \quad (6d < r < 7d) \; . \qquad \text{(P1.158)}$$

The known potential of the third conductor (cylindrical shell) with respect to the ground ($V_3 = 1$ kV) can be expressed in terms of the line integral of **E** from that conductor, to the right, to the fourth conductor, which is grounded ($V_4 = 0$), as follows (Fig.P1.25):

$$V_3 = \int_{r=6d}^{7d} E_3 \, dr = \frac{Q_1' + Q_3'}{2\pi\varepsilon_0} \ln \frac{7d}{6d} = \frac{Q_1' + Q_3'}{2\pi\varepsilon_0} \ln \frac{7}{6} \; . \qquad \text{(P1.159)}$$

On the other side, the same potential can be expressed as the line integral of **E** to the left, to the first conductor, which is also grounded ($V_1 = 0$),

$$V_3 = -\left(\int_{2d}^{3d} E_1 \, dr + \int_{4d}^{5d} E_2 \, dr \right) = -\frac{Q_1'}{2\pi\varepsilon_0} \left(\ln \frac{3d}{2d} + \ln \frac{5d}{4d} \right) = -\frac{Q_1'}{2\pi\varepsilon_0} \ln \frac{15}{8} \; , \qquad \text{(P1.160)}$$

where the voltage between the third and the first conductor is computed as the negative of the voltage between the first and the third one. From Eq.(P1.160), $Q_1' = -2\pi\varepsilon_0 V_3 / \ln(15/8) = -88.5$ nC/m, which is then used with Eq.(P1.159) to obtain $Q_3' = 2\pi\varepsilon_0 V_3 / \ln(7/6) - Q_1' = 450$ nC/m.

PROBLEM 1.80 **Three concentric conductors, one grounded.** Having in mind the field and potential expressions from Problem 1.77, we express the known voltage between the inner and middle conductors in Fig.1.56 in terms of the unknown charge of the inner conductor (Q_1), and then solve for this charge:

$$V_1 - V_2 = \int_{r=a}^{b} \frac{Q_1}{4\pi\varepsilon_0 r^2} \, dr = \frac{Q_1(b-a)}{4\pi\varepsilon_0 ab} \quad \longrightarrow \quad Q_1 = \frac{4\pi\varepsilon_0 ab(V_1 - V_2)}{b-a} = 1.85 \text{ pC} \; . \qquad \text{(P1.161)}$$

Once Q_1 is found, we express the given potential of the middle conductor with respect to the ground (i.e., with respect to the outer conductor, which is grounded) in terms of both Q_1 and the unknown charge of the middle conductor (Q_2), to solve for the latter one as well:

$$V_2 = \int_c^d \frac{Q_1 + Q_2}{4\pi\varepsilon_0 r^2}\,\mathrm{d}r = \frac{(Q_1 + Q_2)(d - c)}{4\pi\varepsilon_0 cd} \quad \longrightarrow \quad Q_2 = \frac{4\pi\varepsilon_0 cdV_2}{d - c} - Q_1 = 24.85\ \mathrm{pC}\,. \quad \text{(P1.162)}$$

PROBLEM 1.81 **Charged metallic foil.** The electric field due to a uniformly charged infinitely large flat metallic foil is the same as the field due to an infinite sheet of charge with the same surface charge density, ρ_s, in free space. The vector $\mathbf{E}$ is thus that shown in Fig.1.15 and its magnitude is given by Eq.(1.64); in particular, $E = \rho_\mathrm{s}/(2\varepsilon_0) = 56.47$ V/m.

PROBLEM 1.82 **Two metallic slabs.** (a) This structure, in the new electrostatic state, is shown in Fig.P1.26. As a result of the electrostatic induction, there is induced surface charge on both surfaces of the newly added (uncharged) metallic slab. The surface charge densities on the two surfaces of the first (charged) slab being $\rho_{\mathrm{s}1} = \rho_{\mathrm{s}2} = 1\ \mu\mathrm{C/m}^2$, the charge density on the first surface of the second slab turns out to be $\rho_{\mathrm{s}3} = -\rho_{\mathrm{s}2} = -1\ \mu\mathrm{C/m}^2$. This can be obtained visualizing the electric-field lines that originate at the positive charges of density $\rho_{\mathrm{s}2}$ on the first slab and terminate at the negative charges of density $\rho_{\mathrm{s}3}$ on the second slab. Alternatively, we can apply Gauss' law to a closed surface (rectangular box) with two sides placed inside the metal of the two slabs, as illustrated in Fig.P1.26, so that $\Psi_E = 0$ [because of Eq.(1.181)], which implies that $Q_S = 0$ as well, and ultimately $\rho_{\mathrm{s}2} + \rho_{\mathrm{s}3} = 0$. Finally, since the second slab is uncharged, $\rho_{\mathrm{s}3} + \rho_{\mathrm{s}4} = 0$, and hence $\rho_{\mathrm{s}4} = -\rho_{\mathrm{s}3} = 1\ \mu\mathrm{C/m}^2$ on its farther surface.

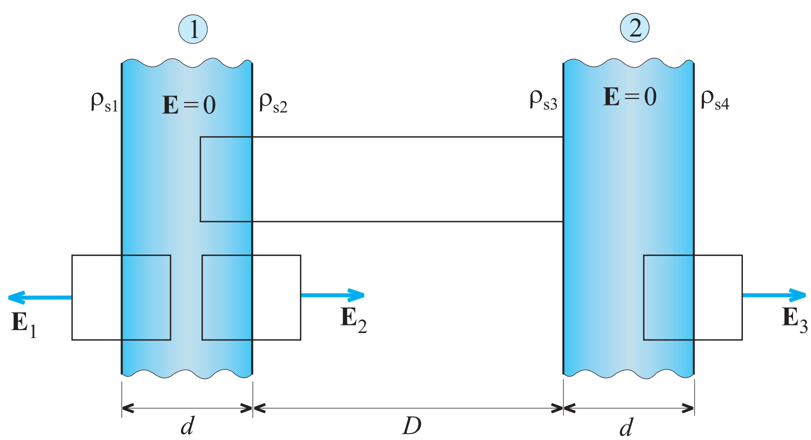

Figure P1.26 Evaluation of the electric field in a system with a charged and an uncharged infinitely large metallic slabs in free space.

(b) Applying Gauss' law to rectangular closed surfaces whose one side is placed inside a metallic slab (where the electric field is zero) and another side parallel to it is in one of the field regions, as indicated in Fig.P1.26, the field intensities in these regions given with respect to the orientations of vectors $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ shown in Fig.P1.26 are $E_1 = \rho_{s1}/(2\varepsilon_0) = 112.94$ kV/m, $E_2 = \rho_{s2}/(2\varepsilon_0) = 112.94$ kV/m, and $E_3 = \rho_{s4}/(2\varepsilon_0) = 112.94$ kV/m. These results can also be obtained by the superposition of fields due four sheets of charge with densities ρ_{s1}, ρ_{s2}, ρ_{s3}, and ρ_{s4} in Fig.P1.26 using Eq.(1.64) for each of the sheets.

(c) The voltage between the slabs amounts to $V = E_2 D = 3.39$ kV.

Section 1.19 Charge Distribution on Metallic Bodies of Arbitrary Shapes

PROBLEM 1.83 Two metallic spheres at the same potential. Since $d/a = 20$ and $d/b = 100$, we conclude that the condition $d \gg a, b$ in Fig.1.45 is satisfied, so we can use equations derived in Section 1.19.

(a) Based on Eq.(1.208), total charges of the two spheres in Fig.1.45 come out to be

$$Q = Q_a + Q_b \quad \text{and} \quad \frac{Q_a}{Q_b} = \frac{a}{b} \quad \longrightarrow \quad Q_a = \frac{a}{a+b} Q = 500 \text{ pC}$$
$$\text{and} \quad Q_b = \frac{b}{a+b} Q = 100 \text{ pC} . \tag{P1.163}$$

Hence, the potential of the spheres, Eqs.(1.206), amounts to

$$V_a = V_b = \frac{Q_a}{4\pi\varepsilon_0 a} = 90 \text{ V} . \tag{P1.164}$$

(b) From Eq.(1.193), the respective electric field intensities near the surfaces of the spheres (Fig.1.45) are given by

$$E_a = \frac{\rho_{sa}}{\varepsilon_0} = \frac{Q_a}{4\pi\varepsilon_0 a^2} = 1.8 \text{ kV/m} \quad \text{and} \quad E_b = \frac{\rho_{sb}}{\varepsilon_0} = \frac{Q_b}{4\pi\varepsilon_0 b^2} = 9 \text{ kV/m} . \tag{P1.165}$$

Note that $E_b = 5E_a$, which is in agreement with Eq.(1.210).

Section 1.20 Method of Moments for Numerical Analysis of Charged Metallic Bodies

PROBLEM 1.84 MoM-based computer program for a charged plate. Computer program for determining the charge distribution of the plate, using the method of moments as presented in Section 1.21, is given in the associated MATLAB exercise.

(a) Results for the surface charge density of $N = 100$ patches in the tabulated form and as a 3-D plot are shown in Fig.P1.27.

ρ_s [pC/m^2]

100.3	66.4	60.7	57.9	56.8	56.8	57.9	60.7	66.4	100.3
66.4	34.1	30.0	28.1	27.4	27.4	28.1	30.0	34.1	66.4
60.7	30.0	25.9	24.0	23.2	23.2	24.0	25.9	30.0	60.7
57.9	28.1	24.0	22.1	21.3	21.3	22.1	24.0	28.1	57.9
56.8	27.4	23.2	21.3	20.6	20.6	21.3	23.2	27.4	56.8
56.8	27.4	23.2	21.3	20.6	20.6	21.3	23.2	27.4	56.8
57.9	28.1	24.0	22.1	21.3	21.3	22.1	24.0	28.1	57.9
60.7	30.0	25.9	24.0	23.2	23.2	24.0	25.9	30.0	60.7
66.4	34.1	30.0	28.1	27.4	27.4	28.1	30.0	34.1	66.4
100.3	66.4	60.7	57.9	56.8	56.8	57.9	60.7	66.4	100.3

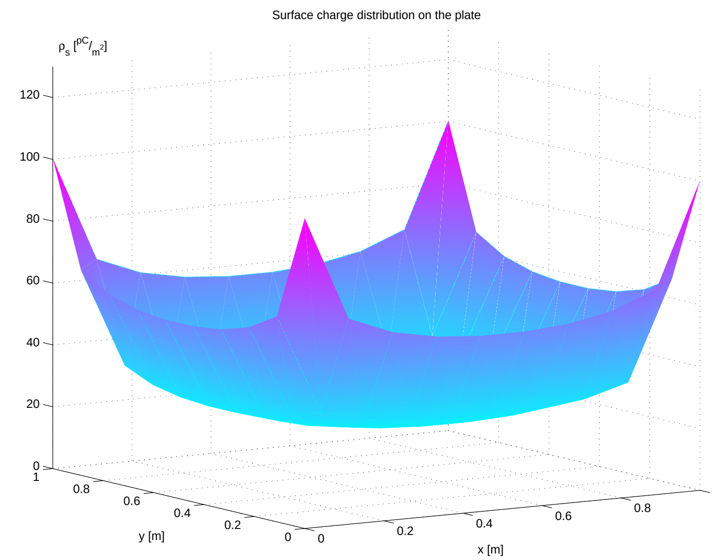

Figure P1.27 Surface charge density (ρ_s) of $N = 100$ (ten partitions in each dimension) patches modeling a thin charged square plate in free space – results obtained using the method of moments (MoM) as presented in Section 1.21 (MoM computer program is given in the associated MATLAB exercise).

(b) The total charge of the plate [Eq.(1.219)], taking (i) $N = 9$, (ii) $N = 25$, (iii) $N = 49$, and (iv) $N = 100$, amounts to (i) $Q = 37.29$ pC, (ii) $Q = 38.71$ pC, (iii) $Q = 39.33$ pC, and (iv) $Q = 39.8$ pC, respectively.

PROBLEM 1.85 **MoM computation for a charged cube.** Computer program for the method-of-moments analysis of the charged metallic cube (Fig.1.46) is given in the associated MATLAB exercise. The calculated total charge of the cube adopting $N = 600$ (ten subdivisions per cube edge) equals $Q = 73.27$ pC.

PROBLEM 1.86 **Approximate integral expression for the electric field vector.** (a) We start with the surface integral expression in Eq.(1.38), for computing, by integration, the electric field intensity vector at an arbitrary point of space due to a surface charge distribution, with density ρ_s, over a surface S, which in our case is the surface of a charged metallic body (as in Fig.1.46). We (exactly or approximately) represent the surface S by N small patches ΔS_i, $i = 1, 2, \ldots, N$ (Fig.1.46), and approximate ρ_s on the ith patch by a (known or unknown) constant ρ_{si}, as in Eq.(1.212). We then approximate the field integral in Eq.(1.38) in the same way the potential integral in Eq.(1.211) is reduced to its approximate form in

Eq.(1.213), based on the charge density approximation in Eq.(1.212), as follows:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_s \, dS}{R^2} \, \hat{\mathbf{R}} \approx \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \rho_{si} \int_{\Delta S_i} \frac{dS}{R^2} \, \hat{\mathbf{R}} \, . \tag{P1.166}$$

The such obtained integrals over individual patches, ΔS_i, can be evaluated analytically (exactly) for some shapes of patches or numerically (approximately) for arbitrary surface elements. However, the simplest way (and the crudest approximation) to compute these surface integrals implies approximating the integrand, $\hat{\mathbf{R}}/R^2$, by its value at the center of the patch, which results in

$$\mathbf{E} \approx \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{\rho_{si}}{R_i^2} \, \hat{\mathbf{R}}_i \int_{\Delta S_i} dS = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{\rho_{si} \Delta S_i}{R_i^2} \, \hat{\mathbf{R}}_i = \sum_{i=1}^{N} \frac{Q_i}{4\pi\varepsilon_0 R_i^2} \, \hat{\mathbf{R}}_i \, ,$$

$$\hat{\mathbf{R}}_i = \frac{\mathbf{R}_i}{R_i} \quad (i = 1, 2, \ldots, N) \, . \tag{P1.167}$$

Essentially, the integrals are evaluated by approximating the uniformly charged patch ΔS_i by an equivalent point charge, $\Delta Q_i = \rho_{si} \Delta S_i$, placed at the patch center, and using the expression for the electric field vector due to a point charge in free space, Eq.(1.24) or (1.25). In this, R_i is the distance of the observation (field) point (at which $\mathbf{E}$ is computed) from the center of the ith patch ($i = 1, 2, \ldots, N$) and $\hat{\mathbf{R}}_i$ is the corresponding unit vector, along the position vector $\mathbf{R}_i$ of the field point with respect to the patch center.

(b) Computer program based on Eq.(P1.167) is provided in the associated MATLAB exercise. The computed electric field intensity along the axis of the plate (from Problem 1.84) perpendicular to its plane at points that are $a/2$, $2a$, and $100a$ distant from the plate surface (for $N = 100$) amounts to $E = 635.5$ mV/m, $E = 82.8$ mV/m, and $E = 35.77$ μV/m, respectively, with the vector $\mathbf{E}$ being along the axis.

(c) For the Cartesian coordinate system adopted as in Fig.1.2, the electric field vector inside the cube (from the previous problem – Problem 1.85), at a quarter of its space diagonal (body diagonal) – point defined by $x = y = z = a/4$, is obtained to be $\mathbf{E} = 20\,(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})$ mV/m, while $\mathbf{E} = 0$ at the cube center.

Section 1.21 Image Theory

PROBLEM 1.87 **Force on a point charge due to its image.** This is similar to the computation in Eqs.(1.223) and (1.224). With reference to Fig.P1.28, the electric force $\mathbf{F}_e$ on the point charge Q in Fig.1.48(a) is given by

$$\mathbf{F}_e = Q\mathbf{E}_{image} = Q \frac{Q}{4\pi\varepsilon_0 (2h)^2} = \frac{Q^2}{16\pi\varepsilon_0 h^2} \quad \text{(force is attractive)} \, . \tag{P1.168}$$

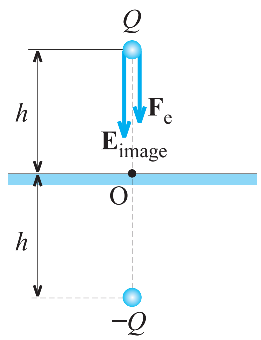

Figure P1.28 Finding the electric force on a point charge Q above a conducting plane, in Fig.1.48(a), using image theory, as in Fig.1.48(b).

PROBLEM 1.88 **Imaging a line charge.** This structure is shown in Fig.P1.29(a). By virtue of image theory, Fig.P1.29(b), the resultant electric field intensity vector at the point M defined by a distance x from the projection of Q' on the symmetry plane (point O) is determined, similarly to the computation in Eqs.(1.220), as

$$\mathbf{E} = \mathbf{E}_{\text{original}} + \mathbf{E}_{\text{image}} = 2E_{Q'} \cos\alpha\,(-\hat{\mathbf{n}}) ,$$

$$E_{Q'} = \frac{Q'}{2\pi\varepsilon_0 R} , \quad R = \sqrt{x^2 + h^2} , \quad \cos\alpha = \frac{h}{R} . \tag{P1.169}$$

Using Eq.(1.190), the surface charge density at the point M on the conducting plane in Fig.P1.29(a) amounts to [also see Eq.(1.221)]

$$\rho_s(x) = \varepsilon_0 \hat{\mathbf{n}} \cdot \mathbf{E} = -\frac{Q'h}{\pi(x^2 + h^2)} . \tag{P1.170}$$

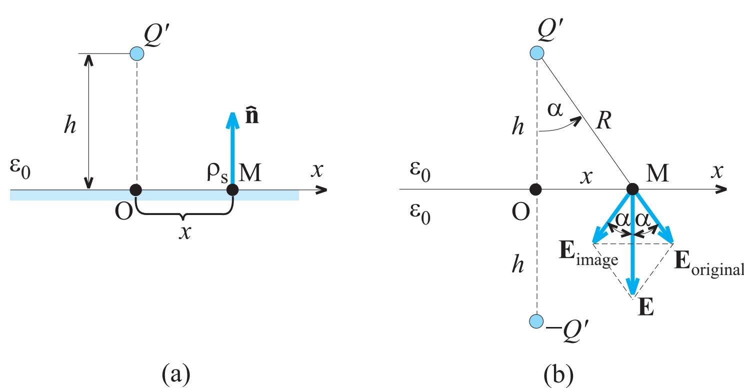

Figure P1.29 Computing the distribution of induced surface charges on a conducting plane underneath a line charge (Fig.1.49): (a) original structure and (b) equivalent structure using image theory.

PROBLEM 1.89 **Charged wire parallel to a corner screen.** We first eliminate the horizontal metallic plane in Fig.1.57 utilizing image theory, which leaves us with two charged wires (line charges), of per-unit-length charges Q' and $-Q'$, respectively, on the right of the vertical metallic plane, which is then removed as well by another application of image theory. The result is a structure with four charged wires, as shown in Fig.P1.30.

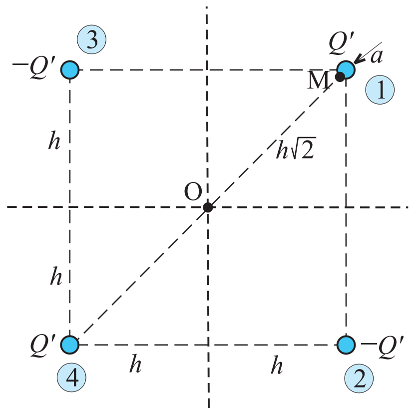

Figure P1.30 Cross section of a system of four charged metallic wires (line charges) in free space equivalent, by virtue of image theory – applied twice, to the structure with one charged wire parallel to a metallic corner screen in Fig.1.57.

To find the voltage between the wire and the screen in Fig.1.57, that is, the potential of the wire with respect to the screen, which is at a zero potential, we use Eq.(1.119) twice, for the two pairs of line charges (wires) in Fig.P1.30. More precisely, we compute the potential at the surface (point M) of the original wire (wire 1). First, for the pair of wires 1 and 2, we have $r_1 = a$ (distance of the point M form the axis of that same wire equals the wire radius) and $r_2 = 2h$ (since $2h \gg a$) in Eq.(1.119), which leads to

$$V_{\text{due to wires 1 and 2}} = \frac{Q'}{2\pi\varepsilon_0} \ln \frac{r_2}{r_1} = \frac{Q'}{2\pi\varepsilon_0} \ln \frac{2h}{a} \quad \text{(on the surface of wire 1)} . \tag{P1.171}$$

Then, we consider the pair of wires 3 and 4 in Fig.P1.30, for which $r_1 = 2h$ (distance between the point M, on the surface of wire 1, from the axis of wire 3) and $r_2 = 2h\sqrt{2}$ (diagonal distance from wire 4) in Eq.(1.119), where also Q' must be switched to $-Q'$ (the upper wire is now charged with $-Q'$), and the corresponding potential is

$$V_{\text{due to wires 3 and 4}} = \frac{-Q'}{2\pi\varepsilon_0} \ln \frac{2h\sqrt{2}}{2h} = -\frac{Q'}{2\pi\varepsilon_0} \ln \sqrt{2} . \tag{P1.172}$$

The resultant potential due to all charged wires (i.e., the voltage we seek) amounts to

$$V = V_{\text{due to wires 1 and 2}} + V_{\text{due to wires 3 and 4}} = \frac{Q'}{2\pi\varepsilon_0} \ln \frac{h\sqrt{2}}{a} . \tag{P1.173}$$

Of course, the same result for V is obtained by the superposition of potentials due to each of the four charged wires considered independently, which, in turn, are calculated employing Eq.(1.87), as is done in deriving Eq.(1.119) – in the first place. Taking the center of the structure (point O) in Fig.P1.30 for the reference point for potential, we thus have

$$V = \frac{Q'}{2\pi\varepsilon_0}\ln\frac{r_{\mathcal{R}1}}{r_1} + \frac{-Q'}{2\pi\varepsilon_0}\ln\frac{r_{\mathcal{R}2}}{r_2} + \frac{-Q'}{2\pi\varepsilon_0}\ln\frac{r_{\mathcal{R}3}}{r_3} + \frac{Q'}{2\pi\varepsilon_0}\ln\frac{r_{\mathcal{R}4}}{r_4}$$

$$= \frac{Q'}{2\pi\varepsilon_0}\ln\frac{h\sqrt{2}}{a} + \frac{-Q'}{2\pi\varepsilon_0}\ln\frac{h\sqrt{2}}{2h} + \frac{-Q'}{2\pi\varepsilon_0}\ln\frac{h\sqrt{2}}{2h} + \frac{Q'}{2\pi\varepsilon_0}\ln\frac{h\sqrt{2}}{2h\sqrt{2}} = \frac{Q'}{2\pi\varepsilon_0}\ln\frac{h\sqrt{2}}{a}\,, \tag{P1.174}$$

where $r_{\mathcal{R}1} = r_{\mathcal{R}2} = r_{\mathcal{R}3} = r_{\mathcal{R}4} = h\sqrt{2}$ are the distances (all the same) of the reference point from individual wires.