Chapter One

1. $AC\{D,B\} = ACDB + ACBD$, $A\{C,B\}D = ACBD + ABCD$, $C\{D,A\}B = CDAB + CADB$, and $\{C,A\}DB = CADB + ACDB$. Therefore $-AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB = -ACDB + ABCD - CDAB + ACDB = ABCD - CDAB = [AB, CD]$

In preparing this solution manual, I have realized that problems 2 and 3 in are misplaced in this chapter. They belong in Chapter Three. The Pauli matrices are not even defined in Chapter One, nor is the math used in previous solution manual. – *Jim Napolitano*

2. (a) $\mathsf{Tr}(X) = a_0\mathsf{Tr}(1) + \sum_\ell \mathsf{Tr}(\sigma_\ell)a_\ell = 2a_0$ since $\mathsf{Tr}(\sigma_\ell) = 0$. Also $\mathsf{Tr}(\sigma_k X) = a_0\mathsf{Tr}(\sigma_k) + \sum_\ell \mathsf{Tr}(\sigma_k\sigma_\ell)a_\ell = \frac{1}{2}\sum_\ell \mathsf{Tr}(\sigma_k\sigma_\ell + \sigma_\ell\sigma_k)a_\ell = \sum_\ell \delta_{k\ell}\mathsf{Tr}(1)a_\ell = 2a_k$. So, $a_0 = \frac{1}{2}\mathsf{Tr}(X)$ and $a_k = \frac{1}{2}\mathsf{Tr}(\sigma_k X)$. (b) Just do the algebra to find $a_0 = (X_{11} + X_{22})/2$, $a_1 = (X_{12} + X_{21})/2$, $a_2 = i(-X_{21} + X_{12})/2$, and $a_3 = (X_{11} - X_{22})/2$.

3. Since $\det(\boldsymbol{\sigma}\cdot\mathbf{a}) = -a_z^2 - (a_x^2 + a_y^2) = -|\mathbf{a}|^2$, the cognoscenti realize that this problem really has to do with rotation operators. From this result, and (3.2.44), we write

$$\det\left[\exp\left(\pm\frac{i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\phi}{2}\right)\right] = \cos\left(\frac{\phi}{2}\right) \pm i\sin\left(\frac{\phi}{2}\right)$$

and multiplying out determinants makes it clear that $\det(\boldsymbol{\sigma}\cdot\mathbf{a}') = \det(\boldsymbol{\sigma}\cdot\mathbf{a})$. Similarly, use (3.2.44) to explicitly write out the matrix $\boldsymbol{\sigma}\cdot\mathbf{a}'$ and equate the elements to those of $\boldsymbol{\sigma}\cdot\mathbf{a}$. With $\hat{\mathbf{n}}$ in the z-direction, it is clear that we have just performed a rotation (of the spin vector) through the angle ϕ.

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4. (a) $\mathsf{Tr}(XY) \equiv \sum_a \langle a|XY|a\rangle = \sum_a\sum_b \langle a|X|b\rangle\langle b|Y|a\rangle$ by inserting the identity operator. Then commute and reverse, so $\mathsf{Tr}(XY) = \sum_b\sum_a \langle b|Y|a\rangle\langle a|X|b\rangle = \sum_b \langle b|YX|b\rangle = \mathsf{Tr}(YX)$.
(b) $XY|\alpha\rangle = X[Y|\alpha\rangle]$ is dual to $\langle\alpha|(XY)^\dagger$, but $Y|\alpha\rangle \equiv |\beta\rangle$ is dual to $\langle\alpha|Y^\dagger \equiv \langle\beta|$ and $X|\beta\rangle$ is dual to $\langle\beta|X^\dagger$ so that $X[Y|\alpha\rangle]$ is dual to $\langle\alpha|Y^\dagger X^\dagger$. Therefore $(XY)^\dagger = Y^\dagger X^\dagger$.
(c) $\exp[if(A)] = \sum_a \exp[if(A)]|a\rangle\langle a| = \sum_a \exp[if(a)]|a\rangle\langle a|$
(d) $\sum_a \psi_a^*(\mathbf{x}')\psi_a(\mathbf{x}'') = \sum_a \langle\mathbf{x}'|a\rangle^*\langle\mathbf{x}''|a\rangle = \sum_a \langle\mathbf{x}''|a\rangle\langle a|\mathbf{x}'\rangle = \langle\mathbf{x}''|\mathbf{x}'\rangle = \delta(\mathbf{x}'' - \mathbf{x}')$

5. For basis kets $|a_i\rangle$, matrix elements of $X \equiv |\alpha\rangle\langle\beta|$ are $X_{ij} = \langle a_i|\alpha\rangle\langle\beta|a_j\rangle = \langle a_i|\alpha\rangle\langle a_j|\beta\rangle^*$. For spin-1/2 in the $|\pm z\rangle$ basis, $\langle +|S_z = \hbar/2\rangle = 1$, $\langle -|S_z = \hbar/2\rangle = 0$, and, using (1.4.17a), $\langle\pm|S_x = \hbar/2\rangle = 1/\sqrt{2}$. Therefore

$$|S_z = \hbar/2\rangle\langle S_x = \hbar/2| \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

6. $A[|i\rangle + |j\rangle] = a_i|i\rangle + a_j|j\rangle \neq [|i\rangle + |j\rangle]$ so in general it is not an eigenvector, unless $a_i = a_j$. That is, $|i\rangle + |j\rangle$ is not an eigenvector of A unless the eigenvalues are degenerate.

7. Since the product is over a complete set, the operator $\prod_{a'}(A-a')$ will always encounter a state $|a_i\rangle$ such that $a'=a_i$ in which case the result is zero. Hence for any state $|\alpha\rangle$

$$\prod_{a'}(A-a')|\alpha\rangle = \prod_{a'}(A-a')\sum_i |a_i\rangle\langle a_i|\alpha\rangle = \sum_i \prod_{a'}(a_i-a')|a_i\rangle\langle a_i|\alpha\rangle = \sum_i 0 = 0$$

If the product instead is over all $a' \neq a_j$ then the only surviving term in the sum is

$$\prod_{a'}(a_j-a')|a_i\rangle\langle a_i|\alpha\rangle$$

and dividing by the factors (a_j-a') just gives the projection of $|\alpha\rangle$ on the direction $|a'\rangle$. For the operator $A \equiv S_z$ and $\{|a'\rangle\} \equiv \{|+\rangle, |-\rangle\}$, we have

$$\begin{aligned}
\prod_{a'}(A-a') &= \left(S_z - \frac{\hbar}{2}\right)\left(S_z + \frac{\hbar}{2}\right) \\
\text{and} \qquad \prod_{a'\neq a''}\frac{A-a'}{a''-a'} &= \frac{S_z+\hbar/2}{\hbar} \qquad \text{for } a'' = +\frac{\hbar}{2} \\
\text{or} \qquad &= \frac{S_z-\hbar/2}{-\hbar} \qquad \text{for } a'' = -\frac{\hbar}{2}
\end{aligned}$$

It is trivial to see that the first operator is the null operator. For the second and third, you can work these out explicitly using (1.3.35) and (1.3.36), for example

$$\frac{S_z+\hbar/2}{\hbar} = \frac{1}{\hbar}\left[S_z + \frac{\hbar}{2}1\right] = \frac{1}{2}\left[(|+\rangle\langle+|) - (|-\rangle\langle-|) + (|+\rangle\langle+|) + (|-\rangle\langle-|)\right] = |+\rangle\langle+|$$

which is just the projection operator for the state $|+\rangle$.

8. I don't see any way to do this problem other than by brute force, and neither did the previous solutions manual. So, make use of $\langle+|+\rangle = 1 = \langle-|-\rangle$ and$\langle+|-\rangle = 0 = \langle-|+\rangle$ and carry through six independent calculations of $[S_i, S_j]$ (along with $[S_i, S_j] = -[S_j, S_i]$) and the six for $\{S_i, S_j\}$ (along with $\{S_i, S_j\} = +\{S_j, S_i\}$).

9. From the figure $\hat{\mathbf{n}} = \hat{\mathbf{i}}\cos\alpha\sin\beta + \hat{\mathbf{j}}\sin\alpha\sin\beta + \hat{\mathbf{k}}\cos\beta$ so we need to find the matrix representation of the operator $\mathbf{S}\cdot\hat{\mathbf{n}} = S_x\cos\alpha\sin\beta + S_y\sin\alpha\sin\beta + S_z\cos\beta$. This means we need the matrix representations of S_x, S_y, and S_z. Get these from the prescription (1.3.19) and the operators represented as outer products in (1.4.18) and (1.3.36), along with the association (1.3.39a) to define which element is which. Thus

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We therefore need to find the (normalized) eigenvector for the matrix

$$\begin{pmatrix} \cos\beta & \cos\alpha\sin\beta - i\sin\alpha\sin\beta \\ \cos\alpha\sin\beta + i\sin\alpha\sin\beta & -\cos\beta \end{pmatrix} = \begin{pmatrix} \cos\beta & e^{-i\alpha}\sin\beta \\ e^{i\alpha}\sin\beta & -\cos\beta \end{pmatrix}$$

with eigenvalue +1. If the upper and lower elements of the eigenvector are a and b, respectively, then we have the equations $|a|^2+|b|^2=1$ and

$$\begin{aligned} a\cos\beta + be^{-i\alpha}\sin\beta &= a \\ ae^{i\alpha}\sin\beta - b\cos\beta &= b \end{aligned}$$

Choose the phase so that a is real and positive. Work with the first equation. (The two equations should be equivalent, since we picked a valid eigenvalue. You should check.) Then

$$\begin{aligned} a^2(1-\cos\beta)^2 &= |b|^2\sin^2\beta = (1-a^2)\sin^2\beta \\ 4a^2\sin^4(\beta/2) &= (1-a^2)4\sin^2(\beta/2)\cos^2(\beta/2) \\ a^2[\sin^2(\beta/2)+\cos^2(\beta/2)] &= \cos^2(\beta/2) \\ a &= \cos(\beta/2) \\ \text{and so} \qquad b &= ae^{i\alpha}\frac{1-\cos\beta}{\sin\beta} = \cos(\beta/2)e^{i\alpha}\frac{2\sin^2(\beta/2)}{2\sin(\beta/2)\cos(\beta/2)} \\ &= e^{i\alpha}\sin(\beta/2) \end{aligned}$$

which agrees with the answer given in the problem.

10. Use simple matrix techniques for this problem. The matrix representation for H is

$$H \doteq \begin{bmatrix} a & a \\ a & -a \end{bmatrix}$$

Eigenvalues E satisfy $(a-E)(-a-E)-a^2 = -2a^2+E^2 = 0$ or $E=\pm a\sqrt{2}$. Let x_1 and x_2 be the two elements of the eigenvector. For $E=+a\sqrt{2}\equiv E^{(1)}$, $(1-\sqrt{2})x_1^{(1)}+x_2^{(1)}=0$, and for $E=-a\sqrt{2}\equiv E^{(2)}$, $(1+\sqrt{2})x_1^{(2)}+x_2^{(2)}=0$. So the eigenstates are represented by

$$|E^{(1)}\rangle \doteq N^{(1)}\begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix} \qquad \text{and} \qquad |E^{(2)}\rangle \doteq N^{(2)}\begin{bmatrix} -1 \\ \sqrt{2}+1 \end{bmatrix}$$

where ${N^{(1)}}^2 = 1/(4-2\sqrt{2})$ and ${N^{(2)}}^2 = 1/(4+2\sqrt{2})$.

11. It is of course possible to solve this using simple matrix techniques. For example, the characteristic equation and eigenvalues are

$$\begin{aligned} 0 &= (H_{11}-\lambda)(H_{22}-\lambda)-H_{12}^2 \\ \lambda &= \frac{H_{11}+H_{22}}{2} \pm \left[\left(\frac{H_{11}-H_{22}}{2}\right)^2 + H_{12}^2\right]^{1/2} \equiv \lambda_\pm \end{aligned}$$

You can go ahead and solve for the eigenvectors, but it is tedious and messy. However, there is a strong hint given that you can make use of spin algebra to solve this problem, another two-state system. The Hamiltonian can be rewritten as

$$H \doteq A\mathsf{1} + B\sigma_z + C\sigma_x$$

where $A \equiv (H_{11}+H_{22})/2$, $B \equiv (H_{11}-H_{22})/2$, and $C \equiv H_{12}$. The eigenvalues of the first term are both A, and the eigenvalues for the sum of the second and third terms are those of $\pm(2/\hbar)$ times a spin vector multiplied by $\sqrt{B^2+C^2}$. In other words, the eigenvalues of the full Hamiltonian are just $A \pm \sqrt{B^2+C^2}$ in full agreement with what we got with usual matrix techniques, above. From the hint (or Problem 9) the eigenvectors must be

$$|\lambda_+\rangle = \cos\frac{\beta}{2}|1\rangle + \sin\frac{\beta}{2}|2\rangle \qquad \text{and} \qquad |\lambda_-\rangle = -\sin\frac{\beta}{2}|1\rangle + \cos\frac{\beta}{2}|2\rangle$$

where $\alpha = 0$, $\tan\beta = C/B = 2H_{12}/(H_{11}-H_{22})$, and we do $\beta \to \pi - \beta$ to "flip the spin."

12. Using the result of Problem 9, the probability of measuring $+\hbar/2$ is

$$\left|\left[\frac{1}{\sqrt{2}}\langle+| + \frac{1}{\sqrt{2}}\langle-|\right]\left[\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle\right]\right|^2 = \frac{1}{2}\left[\sqrt{\frac{1+\cos\gamma}{2}} + \sqrt{\frac{1-\cos\gamma}{2}}\right]^2 = \frac{1+\sin\gamma}{2}$$

The results for $\gamma = 0$ (i.e. $|+\rangle$), $\gamma = \pi/2$ (i.e. $|S_x+\rangle$), and $\gamma = \pi$ (i.e. $|-\rangle$) are 1/2, 1, and 1/2, as expected. Now $\langle(S_x - \langle S_x\rangle)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$, but $S_x^2 = \hbar^2/4$ from Problem 8 and

$$\begin{aligned}\langle S_x\rangle &= \left[\cos\frac{\gamma}{2}\langle+| + \sin\frac{\gamma}{2}\langle-|\right]\frac{\hbar}{2}\left[|+\rangle\langle-| + |-\rangle\langle+|\right]\left[\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle\right] \\ &= \frac{\hbar}{2}\left[\cos\frac{\gamma}{2}\langle-| + \sin\frac{\gamma}{2}\langle+|\right]\left[\cos\frac{\gamma}{2}|+\rangle + \sin\frac{\gamma}{2}|-\rangle\right] = \hbar\cos\frac{\gamma}{2}\sin\frac{\gamma}{2} = \frac{\hbar}{2}\sin\gamma\end{aligned}$$

so $\langle(S_x - \langle S_x\rangle)^2\rangle = \hbar^2(1-\sin^2\gamma)/4 = \hbar^2\cos^2\gamma/4 = \hbar^2/4, 0, \hbar^2 4$ for $\gamma = 0, \pi/2, \pi$.

13. All atoms are in the state $|+\rangle$ after emerging from the first apparatus. The second apparatus projects out the state $|S_n+\rangle$. That is, it acts as the projection operator

$$|S_n+\rangle\langle S_n+| = \left[\cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2}|-\rangle\right]\left[\cos\frac{\beta}{2}\langle+| + \sin\frac{\beta}{2}\langle-|\right]$$

and the third apparatus projects out $|-\rangle$. Therefore, the probability of measuring $-\hbar/2$ after the third apparatus is

$$P(\beta) = |\langle+|S_n+\rangle\langle S_n+|-\rangle|^2 = \cos^2\frac{\beta}{2}\sin^2\frac{\beta}{2} = \frac{1}{4}\sin^2\beta$$

The maximum transmission is for $\beta = 90°$, when 25% of the atoms make it through.

14. The characteristic equation is $-\lambda^3 - 2(-\lambda)(1/\sqrt{2})^2 = \lambda(1-\lambda^2) = 0$ so the eigenvalues are $\lambda = 0, \pm1$ and there is no degeneracy. The eigenvectors corresponding to these are

$$\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix}1\\\sqrt{2}\\1\end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix}1\\-\sqrt{2}\\1\end{bmatrix}$$

The matrix algebra is not hard, but I did this with MATLAB using

```
M=[[0 1 0];[1 0 1];[0 1 0]]/sqrt(2)
[V,D]=eig(M)
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These are the eigenvectors corresponding to the a spin-one system, for a measurement in the x-direction in terms of a basis defined in the z-direction. I'm not sure if there is enough information in Chapter One, though, in order to deduce this.

15. The answer is *yes*. The identity operator is $1 = \sum_{a',b'} |a', b'\rangle\langle a', b'|$ so

$$AB = AB1 = AB \sum_{a',b'} |a', b'\rangle\langle a', b'| = A \sum_{a',b'} b'|a', b'\rangle\langle a', b'| = \sum_{a',b'} b'a'|a', b'\rangle\langle a', b'| = BA$$

Completeness is powerful. It is important to note that the sum must be over both a' and b' in order to span the complete set of sets.

16. Since $AB = -BA$ and $AB|a, b\rangle = ab|a, b\rangle = BA|a, b\rangle$, we must have $ab = -ba$ where both a and b are real numbers. This can only be satisfied if $a = 0$ or $b = 0$ or both.

17. Assume there is no degeneracy and look for an inconsistency with our assumptions. If $|n\rangle$ is a nondegenerate energy eigenstate with eigenvalue E_n, then it is the *only* state with this energy. Since $[H.A_1] = 0$, we must have $HA_1|n\rangle = A_1H|n\rangle = E_nA_1|n\rangle$. That is, $A_1|n\rangle$ is an eigenstate of energy with eigenvalue E_n. Since H and A_1 commute, though, they may have simultaneous eigenstates. Therefore, $A_1|n\rangle = a_1|n\rangle$ since there is only one energy eigenstate.

Similarly, $A_2|n\rangle$ is also an eigenstate of energy with eigenvalue E_n, and $A_2|n\rangle = a_2|n\rangle$. But $A_1A_2|n\rangle = a_2A_1|n\rangle = a_2a_1|n\rangle$ and $A_2A_1|n\rangle = a_1a_2|n\rangle$, where a_1 and a_2 are real numbers. This cannot be true, in general, if $A_1A_2 \neq A_2A_1$ so our assumption of "no degeneracy" must be wrong. There is an out, though, if $a_1 = 0$ or $a_2 = 0$, since one operator acts on zero.

The example given is from a "central forces" Hamiltonian. (See Chapter Three.) The Hamiltonian commutes with the orbital angular momentum operators L_x and L_y, but $[L_x, L_y] \neq 0$. Therefore, in general, there is a degeneracy in these problems. The degeneracy is avoided, though for S-states, where the quantum numbers of L_x and L_y are both necessarily zero.

18. The positivity postulate says that $\langle\gamma|\gamma\rangle \geq 0$, and we apply this to $|\gamma\rangle \equiv |\alpha\rangle + \lambda|\beta\rangle$. The text shows how to apply this to prove the Schwarz Innequality $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$, from which one derives the generalized uncertainty relation (1.4.53), namely

$$\langle(\Delta A)^2(\Delta B)^2\rangle \geq \frac{1}{4}\left|\langle[A, B]\rangle\right|^2$$

Note that $[\Delta A, \Delta B] = [A - \langle A\rangle, B - \langle B\rangle] = [A, B]$. Taking $\Delta A|\alpha\rangle = \lambda\Delta B|\alpha\rangle$ with $\lambda^* = -\lambda$, as suggested, so $\langle\alpha|\Delta A = -\lambda\langle\alpha|\Delta B$, for a particular state $|\alpha\rangle$. Then

$$\langle\alpha|[A, B]|\alpha\rangle = \langle\alpha|\Delta A\Delta B - \Delta B\Delta A|\alpha\rangle = -2\lambda\langle\alpha|(\Delta B)^2|\alpha\rangle$$

and the equality is clearly satisfied in (1.4.53). We are now asked to verify this relationship for a state $|\alpha\rangle$ that is a gaussian wave packet when expressed as a wave function $\langle x'|\alpha\rangle$. Use

$$\begin{aligned}
\langle x'|\Delta x|\alpha\rangle &= \langle x'|x|\alpha\rangle - \langle x\rangle\langle x'|\alpha\rangle = (x' - \langle x\rangle)\langle x'|\alpha\rangle \\
\text{and} \qquad \langle x'|\Delta p|\alpha\rangle &= \langle x'|p|\alpha\rangle - \langle p\rangle\langle x'|\alpha\rangle = \frac{\hbar}{i}\frac{d}{dx'}\langle x'|\alpha\rangle - \langle p\rangle\langle x'|\alpha\rangle \\
\text{with} \qquad \langle x'|\alpha\rangle &= (2\pi d^2)^{-1/4}\exp\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x'-\langle x\rangle)^2}{4d^2}\right] \\
\text{to get} \qquad \frac{\hbar}{i}\frac{d}{dx'}\langle x'|\alpha\rangle &= \left[\langle p\rangle - \frac{\hbar}{i}\frac{1}{2d^2}(x'-\langle x\rangle)\right]\langle x'|\alpha\rangle \\
\text{and so} \qquad \langle x'|\Delta p|\alpha\rangle &= i\frac{\hbar}{2d^2}(x'-\langle x\rangle)\langle x'|\alpha\rangle = \lambda\langle x'|\Delta x|\alpha\rangle
\end{aligned}$$

where λ is a purely imaginary number. The conjecture is satisfied.

It is very simple to show that this condition is satisfied for the ground state of the harmonic oscillator. Refer to (2.3.24) and (2.3.25). Clearly $\langle x\rangle = 0 = \langle p\rangle$ for any eigenstate $|n\rangle$, and $x|0\rangle$ is proportional to $p|0\rangle$, with a proportionality constant that is purely imaginary.

19. Note the obvious typographical error, i.e. S_2^x should be S_x^2. Have $S_x^2 = \hbar^2/4 = S_y^2 = S_z^2$, also $[S_x, S_y] = i\hbar S_z$, all from Problem 8. Now $\langle S_x\rangle = \langle S_y\rangle = 0$ for the $|+\rangle$ state. Then $\langle(\Delta S_x)^2\rangle = \hbar^2/4 = \langle(\Delta S_y)^2\rangle$, and $\langle(\Delta S_x)^2\langle(\Delta S_y)^2 = \hbar^4/16$. Also $|\langle[S_x, S_y]\rangle|^2/4 = \hbar^2|\langle S_z\rangle|^2/4 = \hbar^4/16$ and the generalized uncertainty principle is satisfied by the equality. On the other hand, for the $|S_x+\rangle$ state, $\langle(\Delta S_x)^2\rangle = 0$ and $\langle S_z\rangle = 0$, and again the generalized uncertainty principle is satisfied with an equality.

20. Refer to Problems 8 and 9. Parameterize the state as $|\rangle = \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle$, so

$$\begin{aligned}
\langle S_x\rangle &= \frac{\hbar}{2}\left[\cos\frac{\beta}{2}\langle+| + e^{-i\alpha}\sin\frac{\beta}{2}\langle-|\right][|+\rangle\langle-| + |-\rangle\langle+|]\left[\cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle\right] \\
&= \frac{\hbar}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}(e^{i\alpha}+e^{-i\alpha}) = \frac{\hbar}{2}\sin\beta\cos\alpha \\
\langle(\Delta S_x)^2\rangle &= \langle S_x^2\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4}(1-\sin^2\beta\cos^2\alpha) \qquad \text{(see prob 12)} \\
\langle S_y\rangle &= i\frac{\hbar}{2}\left[\cos\frac{\beta}{2}\langle+| + e^{-i\alpha}\sin\frac{\beta}{2}\langle-|\right][-|+\rangle\langle-| + |-\rangle\langle+|]\left[\cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle\right] \\
&= i\frac{\hbar}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}(e^{i\alpha}-e^{-i\alpha}) = -\frac{\hbar}{2}\sin\beta\sin\alpha \\
\langle(\Delta S_y)^2\rangle &= \langle S_y^2\rangle - \langle S_y\rangle^2 = \frac{\hbar^2}{4}(1-\sin^2\beta\sin^2\alpha)
\end{aligned}$$

Therefore, the left side of the uncertainty relation is

$$\begin{aligned}\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle &= \frac{\hbar^4}{16}(1-\sin^2\beta\cos^2\alpha)(1-\sin^2\beta\sin^2\alpha) \\ &= \frac{\hbar^4}{16}\left(1-\sin^2\beta+\frac{1}{4}\sin^4\beta\sin^2 2\alpha\right) \\ &= \frac{\hbar^4}{16}\left(\cos^2\beta+\frac{1}{4}\sin^4\beta\sin^2 2\alpha\right) \equiv P(\alpha,\beta)\end{aligned}$$

which is clearly maximized when $\sin 2\alpha = \pm 1$ for any value of β. In other words, the uncertainty product is a maximum when the state is pointing in a direction that is 45° with respect to the x or y axes in any quadrant, for any tilt angle β relative to the z-axis. This makes sense. The maximum tilt angle is derived from

$$\frac{\partial P}{\partial\beta} \propto -2\cos\beta\sin\beta+\sin^3\beta\cos\beta(1) = \cos\beta\sin\beta(-2+\sin^2\beta) = 0$$

or $\sin\beta = \pm 1/\sqrt{2}$, that is, 45° with respect to the z-axis. It all hangs together. The maximum uncertainty product is

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \frac{\hbar^4}{16}\left(\frac{1}{2}+\frac{1}{4}\frac{1}{4}\right) = \frac{9}{256}\hbar^4$$

The right side of the uncertainty relation is $|\langle[S_x,S_y]\rangle|^2/4 = \hbar^2|\langle S_z\rangle|^2/4$, so we also need

$$\langle S_z\rangle = \frac{\hbar}{2}\left[\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right] = \frac{\hbar}{2}\cos\beta$$

so the value of the right hand side at maximum is

$$\frac{\hbar^2}{4}|\langle S_z\rangle|^2 = \frac{\hbar^2}{4}\frac{\hbar^2}{4}\frac{1}{2} = \frac{8}{256}\hbar^4$$

and the uncertainty principle is indeed satisfied.

21. The wave function is $\langle x|n\rangle = \sqrt{2/a}\sin(n\pi x/a)$ for $n = 1,2,3,\ldots$, so we calculate

$$\begin{aligned}\langle x|x|n\rangle &= \int_0^a \langle n|x\rangle x\langle x|n\rangle dx = \frac{a}{2} \\ \langle x|x^2|n\rangle &= \int_0^a \langle n|x\rangle x^2\langle x|n\rangle dx = \frac{a^2}{6}\left(-\frac{3}{n^2\pi^2}+2\right) \\ (\Delta x)^2 &= \frac{a^2}{6}\left(-\frac{3}{n^2\pi^2}+2-\frac{6}{4}\right) = \frac{a^2}{6}\left(-\frac{3}{n^2\pi^2}+\frac{1}{2}\right) \\ \langle x|p|n\rangle &= \int_0^a \langle n|x\rangle\frac{\hbar}{i}\frac{d}{dx}\langle x|n\rangle dx = 0 \\ \langle x|p^2|n\rangle &= -\hbar^2\int_0^a \langle n|x\rangle\frac{d^2}{dx^2}\langle x|n\rangle dx = \frac{n^2\pi^2\hbar^2}{a^2} = (\Delta p)^2\end{aligned}$$

(I did these with MAPLE.) Since $[x,p] = i\hbar$, we compare $(\Delta x)^2(\Delta p)^2$ to $\hbar^2/4$ with

$$(\Delta x)^2(\Delta p)^2 = \frac{\hbar^2}{6}\left(-3 + \frac{n^2\pi^2}{2}\right) = \frac{\hbar^2}{4}\left(\frac{n^2\pi^2}{3} - 2\right)$$

which shows that the uncertainty principle is satisfied, since $n\pi^2/3 > n\pi > 3$ for all n.

22. We're looking for a "rough order of magnitude" estimate, so go crazy with the approximations. Model the ice pick as a mass m and length L, standing vertically on the point, i.e. and inverted pendulum. The angular acceleration is $\ddot{\theta}$, the moment of inertia is mL^2 and the torque is $mgL\sin\theta$ where θ is the angle from the vertical. So $mL^2\ddot{\theta} = mgL\sin\theta$ or $\ddot{\theta} = \sqrt{g/L}\sin\theta$. Since $\theta \ll 0$ as the pick starts to fall, take $\sin\theta = \theta$ so

$$\begin{aligned} \theta(t) &= A\exp\left(\sqrt{\frac{g}{L}}t\right) + B\exp\left(-\sqrt{\frac{g}{L}}t\right) \\ x_0 \equiv \theta(0)L &= (A+B)L \\ p_0 \equiv m\dot{\theta}(0)L &= m\sqrt{\frac{g}{L}}(A-B)L = \sqrt{m^2gL}(A-B) \end{aligned}$$

Let the uncertainty principle relate x_0 and p_0, i.e. $x_0p_0 = \sqrt{m^2gL^3}(A^2 - B^2) = \hbar$. Now ignore B; the exponential decay will become irrelevant quickly. You can notice that the pick is falling when it is tilting by something like $1^\circ = \pi/180$, so solve for a time T where $\theta(T) = \pi/180$. Then

$$T = \sqrt{\frac{L}{g}}\ln\frac{\pi/180}{A} = \sqrt{\frac{L}{g}}\left(\frac{1}{4}\ln\frac{m^2gL^3}{\hbar^2} - \ln\frac{180}{\pi}\right)$$

Take $L = 10$ cm, so $\sqrt{L/g} \approx 0.1$ sec, but the action is in the logarithms. (It is worth your time to confirm that the argument of the logarithm in the first term is indeed dimensionless.) Now $\ln(180/\pi) \approx 4$ but the first term appears to be much larger. This is good, since it means that quantum mechanics is driving the result. For $m = 0.1$ kg, find $m^2gL^3/\hbar^2 = 10^{64}$, and so $T = 0.1 \text{ sec} \times (147/4 - 4) \sim 3$ sec. I'd say that's a surprising and interesting result.

23. The eigenvalues of A are obviously $\pm a$, with $-a$ twice. The characteristic equation for B is $(b-\lambda)(-\lambda)^2 - (b-\lambda)(ib)(-ib) = (b-\lambda)(\lambda^2 - b^2) = 0$, so its eigenvalues are $\pm b$ with b twice. (Yes, B has degenerate eigenvalues.) It is easy enough to show that

$$AB = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} = BA$$

so A and B commute, and therefore must have simultaneous eigenvectors. To find these, write the eigenvector components as u_i, $i = 1,2,3$. Clearly, the basis states $|1\rangle$, $|2\rangle$, and $|3\rangle$ are eigenvectors of A with eigenvalues a, $-a$, and $-a$ respectively. So, do the math to find the eigenvectors for B in this basis. Presumably, some freedom will appear that allows us to linear combinations that are also eigenvectors of A. One of these is obviously $|1\rangle \equiv |a, b\rangle$, so just work with the reduced 2×2 basis of states $|2\rangle$ and $|3\rangle$. Indeed, both of these states have eigenvalues a for A, so one linear combinations should have eigenvalue $+b$ for B, and orthogonal combination with eigenvalue $-b$.

Let the eigenvector components be u_2 and u_3. Then, for eigenvalue $+b$,

$$-ibu_3 = +bu_2 \qquad \text{and} \qquad ibu_2 = +bu_3$$

both of which imply $u_3 = iu_2$. For eigenvalue $-b$,

$$-ibu_3 = -bu_2 \qquad \text{and} \qquad ibu_2 = -bu_3$$

both of which imply $u_3 = -iu_2$. Choosing u_2 to be real, then ("No, the eigenvalue alone does not completely characterize the eigenket.") we have the set of simultaneous eigenstates

Eigenvalue of		
A	B	Eigenstate
a	b	$\|1\rangle$
$-a$	b	$\frac{1}{\sqrt{2}}(\|2\rangle + i\|3\rangle)$
$-a$	$-b$	$\frac{1}{\sqrt{2}}(\|2\rangle - i\|3\rangle)$

24. *This problem also appears to belong in Chapter Three. The Pauli matrices are not defined in Chapter One, but perhaps one could simply define these matrices, here and in Problems 2 and 3.*
Operating on the spinor representation of $|+\rangle$ with $(1\sqrt{2})(1 + i\sigma_x)$ gives

$$\frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

So, for an operator U such that $U \doteq (1\sqrt{2})(1+i\sigma_x)$, we observe that $U|+\rangle = |S_y;+\rangle$, defined in (1.4.17b). Similarly operating on the spinor representation of $|-\rangle$ gives

$$\frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

that is, $U|-\rangle = i|S_y;-\rangle$. This is what we would mean by a "rotation" about the x-axis by $90°$. The sense of the rotation is about the $+x$ direction vector, so this would actually be a rotation of $-\pi/2$. (See the diagram following Problem Nine.) The phase factor $i = e^{i\pi/2}$ does not affect this conclusions, and in fact leads to observable quantum mechanical effects. (This is all discussed in Chapter Three.) The matrix elements of S_z in the S_y basis are then

$$\begin{aligned}
\langle S_y;+|S_z|S_y;+\rangle &= \langle +|U^\dagger S_z U|+\rangle \\
\langle S_y;+|S_z|S_y;-\rangle &= -i\langle +|U^\dagger S_z U|-\rangle \\
\langle S_y;-|S_z|S_y;+\rangle &= i\langle -|U^\dagger S_z U|+\rangle \\
\langle S_y;-|S_z|S_y;-\rangle &= \langle -|U^\dagger S_z U|-\rangle
\end{aligned}$$

Note that $\sigma_x^\dagger = \sigma_x$ and $\sigma_x^2 = 1$, so $U^\dagger U \doteq (1\sqrt{2})(1 - i\sigma_x)(1\sqrt{2})(1 + i\sigma_x) = (1/2)(1 + \sigma_x^2) = 1$ and U is therefore unitary. (This is no accident, as will be discussed when rotation operators are presented in Chapter Three.) Furthermore $\sigma_z\sigma_x = -\sigma_x\sigma_z$, so

$$\begin{aligned} U^\dagger S_z U &\doteq \frac{1}{\sqrt{2}}(1 - i\sigma_x)\frac{\hbar}{2}\sigma_z\frac{1}{\sqrt{2}}(1 + i\sigma_x) = \frac{\hbar}{2}\frac{1}{2}(1 - i\sigma_x)^2\sigma_z = -i\frac{\hbar}{2}\sigma_x\sigma_z \\ &= -i\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\ \text{so} \qquad S_z &\doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x \end{aligned}$$

in the $|S_y; \pm\rangle$ *basis.* This can be easily checked directly with (1.4.17b), that is

$$S_z|S_y; \pm\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}[|+\rangle \mp i|-\rangle = \frac{\hbar}{2}|S_y; \mp\rangle$$

There seems to be a mistake in the old solution manual, finding $S_z = (\hbar/2)\sigma_y$ instead of σ_x.

25. Transforming to another representation, say the basis $|c\rangle$, we carry out the calculation

$$\langle c'|A|c''\rangle = \sum_{b'}\sum_{b''}\langle c'|b'\rangle\langle b'|A|b''\rangle\langle b''|c''\rangle$$

There is no principle which says that the $\langle c'|b'\rangle$ need to be real, so $\langle c'|A|c''\rangle$ is not necessarily real if $\langle b'|A|b''\rangle$ is real. The problem alludes to Problem 24 as an example, but not that specific question (assuming my solution is correct.) Still, it is obvious, for example, that the operator S_y is "real" in the $|S_y; \pm\rangle$ basis, but is not in the $|\pm\rangle$ basis.

For another example, also suggested in the text, if you calculate

$$\langle p'|x|p''\rangle = \int \langle p'|x|x'\rangle\langle x'|p''\rangle dx' = \int x'\langle p'|x'\rangle\langle x'|p''\rangle dx' = \frac{1}{2\pi\hbar}\int x' e^{i(p''-p')x'/\hbar}dx'$$

and then define $q \equiv p'' - p'$ and $y \equiv x'/\hbar$, then

$$\langle p'|x|p''\rangle = \frac{\hbar}{2\pi i}\frac{d}{dq}\int e^{iqy}dy = \frac{\hbar}{i}\frac{d}{dq}\delta(q)$$

so you can also see that although x is real in the $|x'\rangle$ basis, it is not so in the $|p'\rangle$ basis.

26. From (1.4.17a), $|S_x; \pm\rangle = (|+\rangle \pm |-\rangle)/\sqrt{2}$, so clearly

$$\begin{aligned} U \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}[1\ 0] + \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}[0\ 1] \\ \Longrightarrow \quad &= |S_x : +\rangle\langle +| + |S_x : -\rangle\langle -| \doteq \sum_r |b^{(r)}\rangle\langle a^{(r)}| \end{aligned}$$

27. The idea here is simple. Just insert a complete set of states. Firstly,

$$\langle b''|f(A)|b'\rangle = \sum_{a'} \langle b''|f(A)|a'\rangle\langle a'|b'\rangle = \sum_{a'} f(a')\langle b''|a'\rangle\langle a'|b'\rangle$$

The numbers $\langle a'|b'\rangle$ (and $\langle b''|a'\rangle$) constitute the "transformation matrix" between the two sets of basis states. Similarly for the continuum case,

$$\begin{aligned}\langle \mathbf{p}''|F(r)|\mathbf{p}'\rangle &= \int \langle \mathbf{p}''|F(r)|\mathbf{x}'\rangle\langle \mathbf{x}'|\mathbf{p}'\rangle d^3x' = \int F(r')\langle \mathbf{p}''|\mathbf{x}'\rangle\langle \mathbf{x}'|\mathbf{p}'\rangle d^3x' \\ &= \frac{1}{(2\pi\hbar)^3}\int F(r')e^{i(\mathbf{p}'-\mathbf{p}'')\cdot\mathbf{x}'/\hbar}d^3x'\end{aligned}$$

The angular parts of the integral can be done explicitly. Let $\mathbf{q} \equiv \mathbf{p}' - \mathbf{p}''$ define the "z"-direction. Then

$$\begin{aligned}\langle \mathbf{p}''|F(r)|\mathbf{p}'\rangle &= \frac{2\pi}{(2\pi\hbar)^3}\int dr' F(r')\int_0^\pi \sin\theta d\theta e^{iqr'\cos\theta/\hbar} = \frac{1}{4\pi^2\hbar^3}\int dr' F(r')\int_{-1}^1 d\mu\ e^{iqr'\mu/\hbar} \\ &= \frac{1}{4\pi^2\hbar^3}\int dr' F(r')\frac{\hbar}{iqr'}2i\sin(qr'/\hbar) = \frac{1}{2\pi^2\hbar^2}\int dr' F(r')\frac{\sin(qr'/\hbar)}{qr'}\end{aligned}$$

28. For functions $f(q,p)$ and $g(q,p)$, where q and p are conjugate position and momentum, respectively, the Poisson bracket from classical physics is

$$[f,g]_{\text{classical}} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \qquad \text{so} \qquad [x, F(p_x)]_{\text{classical}} = \frac{\partial F}{\partial p_x}$$

Using (1.6.47), then, we have

$$\left[x, \exp\left(\frac{ip_x a}{\hbar}\right)\right] = i\hbar\left[x, \exp\left(\frac{ip_x a}{\hbar}\right)\right]_{\text{classical}} = i\hbar\frac{\partial}{\partial p_x}\exp\left(\frac{ip_x a}{\hbar}\right) = -a\exp\left(\frac{ip_x a}{\hbar}\right)$$

To show that $\exp(ip_x a/\hbar)|x'\rangle$ is an eigenstate of position, act on it with x. So

$$x\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle = \left[\exp\left(\frac{ip_x a}{\hbar}\right)x - a\exp\left(\frac{ip_x a}{\hbar}\right)\right]|x'\rangle = (x'-a)\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle$$

In other words, $\exp(ip_x a/\hbar)|x'\rangle$ is an eigenstate of x with eigenvalue $x' - a$. That is $\exp(ip_x a/\hbar)|x'\rangle$ is the translation operator with $\Delta x' = -a$, but we knew that. See (1.6.36).

29. I wouldn't say this is "easily derived", but it is straightforward. Expressing $G(\mathbf{p})$ as a power series means $G(\mathbf{p}) = \sum_{nm\ell} a_{nm\ell}p_i^n p_j^m p_k^\ell$. Now

$$\begin{aligned}[x_i, p_i^n] = x_i p_i p_i^{n-1} - p_i^n x_i &= i\hbar p_i^{n-1} + p_i x_i p_i^{n-1} - p_i^n x_i \\ &= 2i\hbar p_i^{n-1} + p_i^2 x_i p_i^{n-2} - p_i^n x_i \\ &\cdots \\ &= ni\hbar p_i^{n-1} \\ \text{so} \qquad [x_i, G(\mathbf{p})] &= i\hbar\frac{\partial G}{\partial p_i}\end{aligned}$$

The procedure is essentially identical to prove that $[p_i, F(\mathbf{x})] = -i\hbar\partial F/\partial x_i$. As for

$$[x^2, p^2] = x^2p^2 - p^2x^2 = x^2p^2 - xp^2x + xp^2x - p^2x^2 = x[x,p^2] + [x,p^2]x$$

make use of $[x,p^2] = i\hbar\partial(p^2)/\partial p = 2i\hbar p$ so that $[x^2,p^2] = 2i\hbar(xp+px)$. The classical Poisson bracket is $[x^2,p^2]_{\text{classical}} = (2x)(2p) - 0 = 4xp$ and so $[x^2,p^2] = i\hbar[x^2,p^2]_{\text{classical}}$ when we let the (classical quantities) x and p commute.

30. This is very similar to problem 28. Using problem 29,

$$[x_i, \mathcal{J}(\mathbf{l})] = \left[x_i, \exp\left(\frac{-i\mathbf{p}\cdot\mathbf{l}}{\hbar}\right)\right] = i\hbar\frac{\partial}{\partial p_i}\exp\left(\frac{-i\mathbf{p}\cdot\mathbf{l}}{\hbar}\right) = l_i\exp\left(\frac{-i\mathbf{p}\cdot\mathbf{l}}{\hbar}\right) = l_i\mathcal{J}(\mathbf{l})$$

We can use this result to calculate the expectation value of x_i. First note that

$$\begin{aligned}\mathcal{J}^\dagger(\mathbf{l})\,[x_i, \mathcal{J}(\mathbf{l})] &= \mathcal{J}^\dagger(\mathbf{l})x_i\mathcal{J}(\mathbf{l}) - \mathcal{J}^\dagger(\mathbf{l})\mathcal{J}(\mathbf{l})x_i = \mathcal{J}^\dagger(\mathbf{l})x_i\mathcal{J}(\mathbf{l}) - x_i \\ &= \mathcal{J}^\dagger(\mathbf{l})l_i\mathcal{J}(\mathbf{l}) = l_i\end{aligned}$$

Therefore, under translation,

$$\langle x_i\rangle = \langle\alpha|x_i|\alpha\rangle \to \langle\alpha|\mathcal{J}^\dagger(\mathbf{l})x_i\mathcal{J}(\mathbf{l})|\alpha\rangle = \langle\alpha|\mathcal{J}^\dagger(\mathbf{l})x_i\mathcal{J}(\mathbf{l})|\alpha\rangle = \langle\alpha|(x_i+l_i)|\alpha\rangle = \langle x_i\rangle + l_i$$

which is exactly what you expect from a translation operator.

31. This is a continued rehash of the last few problems. Since $[\mathbf{x}, \mathcal{J}(\mathbf{dx}')] = \mathbf{dx}'$ by (1.6.25), and since $\mathcal{J}^\dagger[\mathbf{x},\mathcal{J}] = \mathcal{J}^\dagger\mathbf{x}\mathcal{J} - \mathbf{x}$, we have $\mathcal{J}^\dagger(\mathbf{dx}')\mathbf{x}\mathcal{J}(\mathbf{dx}') = \mathbf{x} + \mathcal{J}^\dagger(\mathbf{dx}')\mathbf{dx}' = \mathbf{x} + \mathbf{dx}'$ since we only keep the lowest order in $\mathbf{dx}'$. Therefore $\langle\mathbf{x}\rangle \to \langle\mathbf{x}\rangle + \mathbf{dx}'$. Similarly, from (1.6.45), $[\mathbf{p}, \mathcal{J}(\mathbf{dx}')] = 0$, so $\mathcal{J}^\dagger[\mathbf{p},\mathcal{J}] = \mathcal{J}^\dagger\mathbf{p}\mathcal{J} - \mathbf{p} = 0$. That is $\mathcal{J}^\dagger\mathbf{p}\mathcal{J} = \mathbf{p}$ and $\langle\mathbf{p}\rangle \to \langle\mathbf{p}\rangle$.

32. These are all straightforward. In the following, all integrals are taken with limits from $-\infty$ to ∞. One thing to keep in mind is that odd integrands give zero for the integral, so the right change of variables can be very useful. Also recall that $\int \exp(-ax^2)dx = \sqrt{\pi/a}$, and $\int x^2\exp(-ax^2)dx = -(d/da)\int\exp(-ax^2)dx = \sqrt{\pi}/2a^{3/2}$. So, for the x-space case,

$$\begin{aligned}\langle p\rangle &= \int\langle\alpha|x'\rangle\langle x'|p|\alpha\rangle dx' = \int\langle\alpha|x'\rangle\frac{\hbar}{i}\frac{d}{dx'}\langle x'|\alpha\rangle dx' = \frac{1}{d\sqrt{\pi}}\int\hbar k\exp\left(-\frac{x'^2}{d^2}\right)dx' = \hbar k \\ \langle p^2\rangle &= -\hbar^2\int\langle\alpha|x'\rangle\frac{d^2}{dx'^2}\langle x'|\alpha\rangle dx' \\ &= -\frac{\hbar^2}{d\sqrt{\pi}}\int\exp\left(-ikx' - \frac{x'^2}{2d^2}\right)\frac{d}{dx'}\left[\left(ik - \frac{x'}{d^2}\right)\exp\left(ikx' - \frac{x'^2}{2d^2}\right)\right]dx' \\ &= -\frac{\hbar^2}{d\sqrt{\pi}}\int\left[-\frac{1}{d^2} + \left(ik - \frac{x'}{d^2}\right)^2\right]\exp\left(-\frac{x'^2}{d^2}\right)dx' \\ &= \hbar^2\left[\frac{1}{d^2} + k^2\right] - \frac{\hbar^2}{d^5\sqrt{\pi}}\int x'^2\exp\left(-\frac{x'^2}{d^2}\right)dx' = \hbar^2\left[\frac{1}{d^2} + k^2\right] - \frac{\hbar^2}{2d^2} = \frac{\hbar^2}{2d^2} + \hbar^2k^2\end{aligned}$$

Using instead the momentum space wave function (1.7.42), we have

$$\begin{aligned}
\langle p\rangle &= \int \langle\alpha|p|p'\rangle\langle p'|\alpha\rangle dp' = \int p'|\langle p'|\alpha\rangle|^2 dp' \\
&= \frac{d}{\hbar\sqrt{\pi}}\int p' \exp\left[-\frac{(p'-\hbar k)^2 d^2}{\hbar^2}\right] dp' = \frac{d}{\hbar\sqrt{\pi}}\int (q+\hbar k)\exp\left[-\frac{q^2d^2}{\hbar^2}\right] dq = \hbar k \\
\langle p^2\rangle &= \frac{d}{\hbar\sqrt{\pi}}\int (q+\hbar k)^2 \exp\left[-\frac{q^2d^2}{\hbar^2}\right] dq = \frac{d}{\hbar\sqrt{\pi}}\frac{\sqrt{\pi}}{2}\frac{\hbar^3}{d^3} + (\hbar k)^2 = \frac{\hbar^2}{2d^2} + \hbar^2 k^2
\end{aligned}$$

33. I can't help but think this problem can be done by creating a "momentum translation" operator, but instead I will follow the original solution manual. This approach uses the position space representation and Fourier transform to arrive the answer. Start with

$$\begin{aligned}
\langle p'|x|p''\rangle &= \int \langle p'|x|x'\rangle\langle x'|p''\rangle dx' = \int x'\langle p'|x'\rangle\langle x'|p''\rangle dx' \\
&= \frac{1}{2\pi\hbar}\int x' \exp\left[-i\frac{(p'-p'')\cdot x'}{\hbar}\right] dx' = i\frac{\partial}{\partial p'}\frac{1}{2\pi}\int \exp\left[-i\frac{(p'-p'')\cdot x'}{\hbar}\right] dx' \\
&= i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')
\end{aligned}$$

Now find $\langle p'|x|\alpha\rangle$ by inserting a complete set of states $|p''\rangle$, that is

$$\langle p'|x|\alpha\rangle = \int \langle p'|x|p''\rangle\langle p''|\alpha\rangle dp'' = i\hbar\frac{\partial}{\partial p'}\int \delta(p'-p'')\langle p''|\alpha\rangle dp'' = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$$

Given this, the next expression is simple to prove, namely

$$\langle\beta|x|\alpha\rangle = \int dp'\langle\beta|p'\rangle\langle p'|x|\alpha\rangle = \int dp'\phi^*_\beta(p') i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')$$

using the standard definition $\phi_\gamma(p') \equiv \langle p'|\gamma\rangle$.

Certainly the operator $\mathcal{T}(\Xi) \equiv \exp(ix\Xi/\hbar)$ looks like a momentum translation operator. So, we should try to work out $p\mathcal{T}(\Xi)|p'\rangle = p\exp(ix\Xi/\hbar)|p'\rangle$ and see if we get $|p'+\Xi\rangle$. Take a lesson from problem 28, and make use of the result from problem 29, and we have

$$p\mathcal{T}(\Xi)|p'\rangle = \{\mathcal{T}(\Xi)p + [p,\mathcal{T}(\Xi)]\}|p'\rangle = \left\{p'\mathcal{T}(\Xi) - i\hbar\frac{\partial}{\partial x}\mathcal{T}(\Xi)\right\}|p'\rangle = (p'+\Xi)\mathcal{T}(\Xi)|p'\rangle$$

and, indeed, $\mathcal{T}(\Xi)|p'\rangle$ is an eigenstate of p with eigenvalue $p'+\Xi$. In fact, this could have been done first, and then write down the translation operator for infinitesimal momenta, and derive the expression for $\langle p'|x|\alpha\rangle$ the same way as done in the text for infinitesimal spacial translations. (I like this way of wording the problem, and maybe it will be changed in the next edition.)