Chapter #2 Problems

1. A fluid has the following velocity profile in three dimensions:

$$v_r = v_o\left(r^2\theta + \theta^2 z - r^2 z\right) \qquad \mathrm{v}_\theta = \mathrm{v}_\mathrm{o}\left(\theta^2 z - \theta^2 r + \theta^3\right)$$

$$v_z = v_o\left(zr^2 - z^2\theta\right)$$

a) What are the 9 stresses for this fluid?
b) Is the fluid incompressible?

It is easiest to answer part (b) first. We need to calculate the divergence of v,

$$\left(\nabla \bullet \mathbf{v}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(rv_r\right) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}.$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rv_r\right) = v_o\left(3r\theta + \frac{\theta^2 z}{r} - 3rz\right)$$

$$\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = \frac{v_o}{r}\left(2\theta z - 2\theta r + 3\theta^2\right)$$

$$\frac{\partial v_z}{\partial z} = v_o\left(r^2 - 2\theta z\right)$$

$$\left(\nabla \bullet \mathbf{v}\right) = \frac{v_o}{r}\left(3r^2\left(\theta - z\right) + \theta^2\left(z + 3\right) - 2r\theta + r^3 + 2\theta z - 2r\theta z\right)$$

Since $(\nabla \bullet v) \neq 0$ the fluid is compressible. This is so even though there may be some values of r, θ, z for which $\nabla \bullet v = 0$. To be incompressible, the divergence must be zero everywhere.

The 9 stresses can be calculated by referring to Table 2.4. Dealing with the shear stresses first:

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left(r\frac{\partial}{\partial \theta}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)$$

$$-\mu v_o\left(2\theta z - 2r\theta + 3\theta^2 + r + \frac{2\theta z}{r}\right)$$

$$\tau_{z\theta} = \tau_{\theta z} = -\mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) = -\mu v_o\left(\theta^2 + \frac{z^2}{r}\right)$$

$$\tau_{rz} = \tau_{zr} = -\mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) = -\mu v_o\left(\theta^2 - r^2 + 2zr\right)$$

The normal stresses are:

$$\tau_{rr} = -\left(2\mu\frac{\partial v_r}{\partial r} - \left(\frac{2}{3}\mu - \kappa\right)(\nabla \bullet \mathbf{v})\right)$$

$$= -4\mu v_o r(\theta - z) - \left(\frac{2}{3}\mu - \kappa\right)v_o\left(\frac{\theta^2 z}{r} + 3\theta^2 + r\theta + r - 3rz\right)$$

$$\tau_{\theta\theta} = -\left(2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \left(\frac{2}{3}\mu - \kappa\right)(\nabla \bullet \mathbf{v})\right)$$

$$= -2\mu v_o\left(\frac{2\theta z}{r} - 2\theta + \frac{3\theta^2}{r} + r\theta + \frac{\theta^2 z}{r} - rz\right)$$
$$-\left(\frac{2}{3}\mu - \kappa\right)v_o\left(\frac{\theta^2 z}{r} + 3\theta^2 + r\theta + r - 3rz\right)$$

$$\tau_{zz} = -\left(2\mu\frac{\partial v_z}{\partial z} - \left(\frac{2}{3}\mu - \kappa\right)(\nabla \bullet \mathbf{v})\right)$$

$$= -2\mu v_o\left(2zr - 2\theta z\right) - \left(\frac{2}{3}\mu - \kappa\right)v_o\left(\frac{\theta^2 z}{r} + 3\theta^2 + r\theta + r^2 - 3rz\right)$$

2. In 1856 H. Darcy published a paper where he described experiments showing that the flow of fluid through a porous medium was linearly related to the pressure drop across the medium.

$$\vec{\mathbf{v}} = -\frac{k}{\mu}\vec{\nabla}P \qquad \text{Darcy's Law}$$

where k is the permeability of the porous medium. In three dimensions Darcy's Law can be written as:

$$v_x = -\frac{k}{\mu}\left(\frac{\partial P}{\partial x}\right) \qquad v_y = -\frac{k}{\mu}\left(\frac{\partial P}{\partial y}\right) \qquad v_z = -\frac{k}{\mu}\left(\frac{\partial P}{\partial z}\right)$$

Show, that if the fluid is incompressible, the pressure must obey Laplace's Equation ($\nabla^2 \mathrm{P} = 0$).

If the fluid is incompressible then $\nabla \bullet \mathbf{v} = 0$. If we plug in for v using Darcy's Law we have:

$$-\frac{k}{\mu}\vec{\nabla} \bullet \vec{\nabla}P = 0 = -\frac{k}{\mu}\nabla^2 P$$

and Laplace's equation is satisfied. Of course you can do it brute force by expanding the gradient operator, taking the dot product with the velocity, and then plugging in for the pressure components but that is a lot of work. It is easier to just look up the vector identity.