2

Signals and Systems

SIGNALS AND THEIR PROPERTIES

Solution 2.1

(a) $\delta_s(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m, y-n) = \sum_{m=-\infty}^{\infty} \delta(x-m) \cdot \sum_{n=-\infty}^{\infty} \delta(y-n)$, therefore it is a separable signal.

(b) $\delta_l(x, y)$ is separable if $\sin(2\theta) = 0$. In this case, either $\sin\theta = 0$ or $\cos\theta = 0$, $\delta_l(x, y)$ is a product of a constant function in one axis and a 1-D *delta* function in another. But in general, $\delta_l(x, y)$ is not separable.

(c) $e(x, y) = \exp[j2\pi(u_0 x + v_0 y)] = \exp(j2\pi u_0 x) \cdot \exp(j2\pi v_0 y) = e_{1D}(x; u_0) \cdot e_{1D}(y; v_0)$, where $e_{1D}(t; \omega) = \exp(j2\pi\omega t)$. Therefore, $e(x, y)$ is a separable signal.

(d) $s(x, y)$ is a separable signal when $u_0 v_0 = 0$. For example, if $u_0 = 0$, $s(x, y) = \sin(2\pi v_0 y)$ is the product of a constant signal in x and a 1-D sinusoidal signal in y. But in general, when both u_0 and v_0 are nonzero, $s(x, y)$ is not separable.

Solution 2.2

(a) Not periodic. $\delta(x, y)$ is non-zero only when $x = y = 0$.

(b) Periodic.

$$\mathrm{comb}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m, y-n)$$

For arbitrary integers M and N

$$\begin{aligned}
\mathrm{comb}(x+M, y+N) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m+M, y-n+N) \\
&= \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta(x-p, y-q) \ [\text{let } p = m-M, q = n-N] \\
&= \mathrm{comb}(x, y)
\end{aligned}$$

So the smallest period is 1 in both x and y directions.

(c) Periodic. Let $f(x+T_x, y) = f(x, y)$, we have

$$\sin(2\pi x)\cos(4\pi y) = \sin(2\pi(x+T_x))\cos(4\pi y).$$

Solve the above equation, we have $2\pi T_x = 2k\pi$ for arbitrary integer k. So the smallest period for x is $T_{x0} = 1$. Similarly we can find the smallest period for y is $T_{y0} = 1/2$.

(d) Periodic. Let $f(x+T_x, y) = f(x, y)$, we have

$$\sin(2\pi(x+y)) = \sin(2\pi(x+T_x+y)).$$

So the smallest period for x is $T_{x0} = 1$ and the smallest period for y is $T_{y0} = 1$.

(e) Not periodic. We can see this by contradiction. Suppose $f(x, y) = \sin(2\pi(x^2+y^2))$ is periodic, then there exist some T_x such that $f(x+T_x, y) = f(x, y)$:

$$\begin{aligned}\sin(2\pi(x^2+y^2)) &= \sin(2\pi((x+T_x)^2+y^2)) \\ &= \sin(2\pi(x^2+y^2+2xT_x+T_x^2))\end{aligned}$$

In order for the above equation to hold, we must have $2xT_x + T_x^2 = k$ for some integer k. The solution for T_x depends on x. So $f(x, y) = \sin(2\pi(x^2+y^2))$ is not periodic.

(f) Periodic. Let $f_d(m+M, n) = f_d(m, n)$, we have

$$\sin\left(\frac{\pi}{5}m\right)\cos\left(\frac{\pi}{5}n\right) = \sin\left(\frac{\pi}{5}(m+M)\right)\cos\left(\frac{\pi}{5}n\right)$$

Solve for M, we have $M = 10k$ for any integer k. The smallest period for both m and n is 10.

(g) Not periodic. Analog to part (f), by letting $f_d(m+M, n) = f_d(m, n)$, we have

$$\sin\left(\frac{1}{5}m\right)\cos\left(\frac{1}{5}n\right) = \sin\left(\frac{1}{5}(m+M)\right)\cos\left(\frac{1}{5}n\right)$$

The solution for M is $M = 10k\pi$. Since $f_d(m, n)$ is a discrete signal, its period must be an integer if it is periodic. There is no integer k that solves the equality for $M = 10k\pi$ for some M. So, $f_d(m, n) = \sin\left(\frac{1}{5}m\right)\cos\left(\frac{1}{5}n\right)$ is not periodic.

Solution 2.3

(a)

$$\begin{aligned}E_\infty(\delta_s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta_s^2(x, y)\,dxdy \\ &= \lim_{X\to\infty}\lim_{Y\to\infty}\int_{-X}^{X}\int_{-Y}^{Y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m, y-n)\,dxdy \\ &= \lim_{X\to\infty}\lim_{Y\to\infty}(2\lfloor X\rfloor+1)(2\lfloor Y\rfloor+1) \\ &= \infty\end{aligned}$$

where $\lfloor X\rfloor$ is the greatest integer that is smaller than or equal to X.

$$\begin{aligned}
P_\infty(\delta_s) &= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\delta_s^2(x,y)\,dxdy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m,y-n)\,dxdy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{(2\lfloor X\rfloor+1)(2\lfloor Y\rfloor+1)}{4XY} \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\left\{\frac{4\lfloor X\rfloor\lfloor Y\rfloor}{4XY}+\frac{2\lfloor X\rfloor+2\lfloor Y\rfloor}{4XY}+\frac{1}{4XY}\right\} \\
P_\infty(\delta_s) &= 1
\end{aligned}$$

(b)

$$\begin{aligned}
E_\infty(\delta_l) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\delta(x\cos\theta+y\sin\theta-l)|^2\,dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x\cos\theta+y\sin\theta-l)\,dxdy \\
&\overset{①}{=} \begin{cases}\int_{-\infty}^{\infty}\frac{1}{|\sin\theta|}\,dx, & \sin\theta\neq 0\\[2ex] \int_{-\infty}^{\infty}\frac{1}{|\cos\theta|}\,dy, & \cos\theta\neq 0\end{cases} \\
E_\infty(\delta_l) &= \infty
\end{aligned}$$

① comes from the scaling property of the point impulse. The 1-D version of Eq. (2.8) in the text is $\delta(ax)=\frac{1}{|a|}\delta(x)$.

Suppose $\cos\theta\neq 0$,

$$\delta(x\cos\theta+y\sin\theta-l)=\frac{1}{|\cos\theta|}\,\delta\left(x+y\frac{\sin\theta}{\cos\theta}-\frac{l}{\cos\theta}\right)$$

therefore,

$$\int_{-\infty}^{\infty}\delta(x\cos\theta+y\sin\theta-l)dx=\frac{1}{|\cos\theta|}.$$

$$\begin{aligned}
P_\infty(\delta_l) &= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}|\delta(x\cos\theta+y\sin\theta-l)|^2dxdy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\delta(x\cos\theta+y\sin\theta-l)dxdy
\end{aligned}$$

without loss of generality, assume $\theta=0$ and $l=0$, so we have $\sin\theta=0$ and $\cos\theta=1$. Therefore:

$$P_\infty(\delta_l) = \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\delta(x)\,dxdy$$

$$
\begin{aligned}
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-Y}^{Y}\left\{\int_{-X}^{X}\delta(x)dx\right\}dy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-Y}^{Y}1dx \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{2Y}{4XY} \\
&= \lim_{X\to\infty}\frac{1}{2X} \\
P_\infty(\delta_l) &= 0
\end{aligned}
$$

(c)

$$
\begin{aligned}
E_\infty(e) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\exp[j2\pi(u_0x+v_0y)]|^2\,dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}1\,dxdy \\
E_\infty(e) &= \infty
\end{aligned}
$$

$$
\begin{aligned}
P_\infty(e) &= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}|\exp[j2\pi(u_0x+v_0y)]|^2\,dxdy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}1\,dxdy \\
P_\infty(e) &= 1
\end{aligned}
$$

(d)

$$
\begin{aligned}
E_\infty(s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin^2[2\pi(u_0x+v_0y)]\,dxdy \\
&\overset{②}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1-\cos[4\pi(u_0x+v_0y)]}{2}\,dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}\,dxdy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\cos[4\pi(u_0x+v_0y)]}{2}\,dxdy \\
E_\infty(s) &\overset{③}{=} \infty
\end{aligned}
$$

② comes from the trigonometric identity: $\cos(2\theta) = 1 - 2\sin^2(\theta)$.

③ holds because the first integral goes to infinity. The absolute value of the second integral is bounded, although it does not converge as X and Y go to infinity.

$$
\begin{aligned}
P_\infty(s) &= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\sin^2[2\pi(u_0x+v_0y)]\,dxdy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-Y}^{Y}\left\{\int_{-X}^{X}\frac{1-\cos[4\pi(u_0x+v_0y)]}{2}dx\right\}dy
\end{aligned}
$$