Exercises for Chapter 1

Exercises for Section 1.1: Describing a Set

1.1 Only (d) and (e) are sets.

1.2 (a) $A = \{1, 2, 3\} = \{x \in S : x > 0\}$.

(b) $B = \{0, 1, 2, 3\} = \{x \in S : x \geq 0\}$.

(c) $C = \{-2, -1\} = \{x \in S : x < 0\}$.

(d) $D = \{x \in S : |x| \geq 2\}$.

1.3 (a) $|A| = 5$. (b) $|B| = 11$. (c) $|C| = 51$. (d) $|D| = 2$. (e) $|E| = 1$. (f) $|F| = 2$.

1.4 (a) $A = \{n \in \mathbf{Z} : -4 < n \leq 4\} = \{-3, -2, \ldots, 4\}$.

(b) $B = \{n \in \mathbf{Z} : n^2 < 5\} = \{-2, -1, 0, 1, 2\}$.

(c) $C = \{n \in \mathbf{N} : n^3 < 100\} = \{1, 2, 3, 4\}$.

(d) $D = \{x \in \mathbf{R} : x^2 - x = 0\} = \{0, 1\}$.

(e) $E = \{x \in \mathbf{R} : x^2 + 1 = 0\} = \{\} = \emptyset$.

1.5 (a) $A = \{-1, -2, -3, \ldots\} = \{x \in \mathbf{Z} : x \leq -1\}$.

(b) $B = \{-3, -2, \ldots, 3\} = \{x \in \mathbf{Z} : -3 \leq x \leq 3\} = \{x \in \mathbf{Z} : |x| \leq 3\}$.

(c) $C = \{-2, -1, 1, 2\} = \{x \in \mathbf{Z} : -2 \leq x \leq 2, x \neq 0\} = \{x \in \mathbf{Z} : 0 < |x| \leq 2\}$.

1.6 (a) $A = \{2x + 1 : x \in \mathbf{Z}\} = \{\cdots, -5, -3, -1, 1, 3, 5, \cdots\}$.

(b) $B = \{4n : n \in \mathbf{Z}\} = \{\cdots, -8, -4, 0, 4, 8, \cdots\}$.

(c) $C = \{3q + 1 : q \in \mathbf{Z}\} = \{\cdots, -5, -2, 1, 4, 7, \cdots\}$.

1.7 (a) $A = \{\cdots, -4, -1, 2, 5, 8, \cdots\} = \{3x + 2 : x \in \mathbf{Z}\}$.

(b) $B = \{\cdots, -10, -5, 0, 5, 10, \cdots\} = \{5x : x \in \mathbf{Z}\}$.

(c) $C = \{1, 8, 27, 64, 125, \cdots\} = \{x^3 : x \in \mathbf{N}\}$.

1.8 (a) $A = \{n \in \mathbf{Z} : 2 \leq |n| < 4\} = \{-3, -2, 2, 3\}$.

(b) 5/2, 7/2, 4.

(c) $C = \{x \in \mathbf{R} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0\} = \{x \in \mathbf{R} : (x - 2)(x - \sqrt{2}) = 0\} = \{2, \sqrt{2}\}$.

(d) $D = \{x \in \mathbf{Q} : x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0\} = \{2\}$.

(e) $|A| = 4$, $|C| = 2$, $|D| = 1$.

1.9 $A = \{2, 3, 5, 7, 8, 10, 13\}$.

$B = \{x \in A : x = y + z, \text{ where } y, z \in A\} = \{5, 7, 8, 10, 13\}$.

$C = \{r \in B : r + s \in B \text{ for some } s \in B\} = \{5, 8\}$.

Exercises for Section 1.2: Subsets

1.10 (a) $A = \{1, 2\}$, $B = \{1, 2\}$, $C = \{1, 2, 3\}$.

(b) $A = \{1\}$, $B = \{\{1\}, 2\}$. $C = \{\{\{1\}, 2\}, 1\}$.

(c) $A = \{1\}$, $B = \{\{1\}, 2\}$, $C = \{1, 2\}$.

1.11 Let $r = \min(c - a, b - c)$ and let $I = (c - r, c + r)$. Then I is centered at c and $I \subseteq (a, b)$.

1.12 $A = B = D = E = \{-1, 0, 1\}$ and $C = \{0, 1\}$.

1.13 See Figure 1.

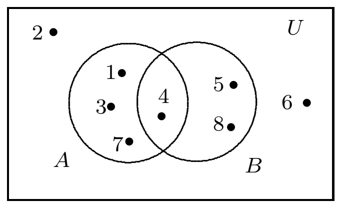

Figure 1: Answer for Exercise 1.13

1.14 (a) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$; $|\mathcal{P}(A)| = 4$.

(b) $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{1\}, \{\{a\}\}, \{\emptyset, 1\}, \{\emptyset, \{a\}\}, \{1, \{a\}\}, \{\emptyset, 1, \{a\}\}\}$; $|\mathcal{P}(A)| = 8$.

1.15 $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\{0\}\}, A\}$.

1.16 $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$, $\mathcal{P}(\mathcal{P}(\{1\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$; $|\mathcal{P}(\mathcal{P}(\{1\}))| = 4$.

1.17 $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\emptyset\}, \{\{\emptyset\}\}, \{0, \emptyset\}, \{0, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, A\}$; $|\mathcal{P}(A)| = 8$.

1.18 $\mathcal{P}(\{0\}) = \{\emptyset, \{0\}\}$.

$A = \{x : x = 0 \text{ or } x \in \mathcal{P}(\{0\})\} = \{0, \emptyset, \{0\}\}$.

$\mathcal{P}(A) = \{\emptyset, \{0\}, \{\emptyset\}, \{\{0\}\}, \{0, \emptyset\}, \{0, \{0\}\}, \{\emptyset, \{0\}\}, A\}$.

1.19 (a) $S = \{\emptyset, \{1\}\}$.

(b) $S = \{1\}$.

(c) $S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4, 5\}\}$.

(d) $S = \{1, 2, 3, 4, 5\}$.

1.20 (a) False. For example, for $A = \{1, \{1\}\}$, both $1 \in A$ and $\{1\} \in A$.

(b) Because $\mathcal{P}(B)$ is the set of all subsets of the set B and $A \subset \mathcal{P}(B)$ with $|A| = 2$, it follows that A is a proper subset of $\mathcal{P}(B)$ consisting of exactly two elements of $\mathcal{P}(B)$. Thus $\mathcal{P}(B)$ contains at least one element that is not in A. Suppose that $|B| = n$. Then $|\mathcal{P}(B)| = 2^n$. Since $2^n > 2$, it follows that $n \geq 2$ and $|\mathcal{P}(B)| = 2^n \geq 4$. Because $\mathcal{P}(B) \subset C$, it is impossible that $|C| = 4$. Suppose that $A = \{\{1\}, \{2\}\}$, $B = \{1, 2\}$ and $C = \mathcal{P}(B) \cup \{3\}$. Then $A \subset \mathcal{P}(B) \subset C$, where $|A| = 2$ and $|C| = 5$.

(c) No. For $A = \emptyset$ and $B = \{1\}$, $|\mathcal{P}(A)| = 1$ and $|\mathcal{P}(B)| = 2$.

(d) Yes. There are only three distinct subsets of $\{1, 2, 3\}$ with two elements.

1.21 $B = \{1, 4, 5\}$.

Exercises for Section 1.3: Set Operations

1.22 (a) $A \cup B = \{1, 3, 5, 9, 13, 15\}$.

(b) $A \cap B = \{9\}$.

(c) $A - B = \{1, 5, 13\}$.

(d) $B - A = \{3, 15\}$.

(e) $\overline{A} = \{3, 7, 11, 15\}$.

(f) $A \cap \overline{B} = \{1, 5, 13\}$.

1.23 Let $A = \{1, 2, \ldots, 6\}$ and $B = \{4, 5, \ldots, 9\}$. Then $A - B = \{1, 2, 3\}$, $B - A = \{7, 8, 9\}$ and $A \cap B = \{4, 5, 6\}$. Thus $|A - B| = |A \cap B| = |B - A| = 3$. See Figure 2.

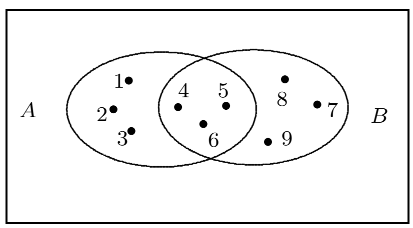

Figure 2: Answer for Exercise 1.23

1.24 Let $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{2, 3\}$. Then $B \neq C$ but $B - A = C - A = \{3\}$.

1.25 (a) $A = \{1\}$, $B = \{\{1\}\}$, $C = \{1, 2\}$.

(b) $A = \{\{1\}, 1\}$, $B = \{1\}$, $C = \{1, 2\}$.

(c) $A = \{1\}$, $B = \{\{1\}\}$, $C = \{\{1\}, 2\}$.

1.26 (a) and (b) are the same, as are (c) and (d).

1.27 Let $U = \{1, 2, \ldots, 8\}$ be a universal set, $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Then $A - B = \{1, 2\}$, $B - A = \{5, 6\}$, $A \cap B = \{3, 4\}$ and $\overline{A \cup B} = \{7, 8\}$. See Figure 3.

1.28 See Figures 4(a) and 4(b).

1.29 (a) The sets $\emptyset$ and $\{\emptyset\}$ are elements of A.

(b) $|A| = 3$.

(c) All of $\emptyset$, $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$ are subsets of A.

(d) $\emptyset \cap A = \emptyset$.

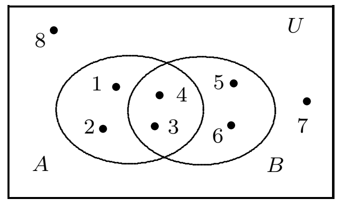

Figure 3: Answer for Exercise 1.27

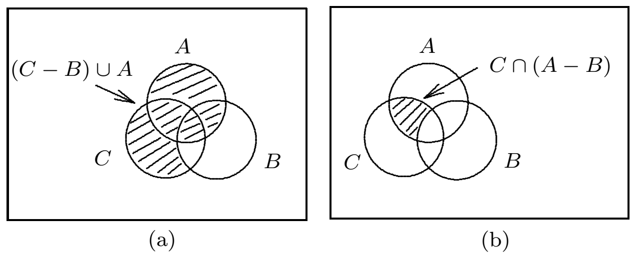

Figure 4: Answers for Exercise 1.28

(e) $\{\emptyset\}\cap A=\{\emptyset\}$.

(f) $\{\emptyset,\{\emptyset\}\}\cap A=\{\emptyset,\{\emptyset\}\}$.

(g) $\emptyset\cup A=A$.

(h) $\{\emptyset\}\cup A=A$.

(i) $\{\emptyset,\{\emptyset\}\}\cup A=A$.

1.30 (a) $A=\{x\in\mathbf{R}:|x-1|\le 2\}=\{x\in\mathbf{R}:-2\le x-1\le 2\}=\{x\in\mathbf{R}:-1\le x\le 3\}=[-1,3]$
$B=\{x\in\mathbf{R}:|x|\ge 1\}=\{x\in\mathbf{R}:x\ge 1 \text{ or } x\le -1\}=(-\infty,-1]\cup[1,\infty)$
$C=\{x\in\mathbf{R}:|x+2|\le 3\}=\{x\in\mathbf{R}:-3\le x+2\le 3\}=\{x\in\mathbf{R}:-5\le x\le 1\}=[-5,1]$

(b) $A\cup B=(-\infty,\infty)=\mathbf{R}$, $A\cap B=\{-1\}\cup[1,3]$,
$B\cap C=[-5,-1]\cup\{1\}$, $B-C=(-\infty,-5)\cup(1,\infty)$.

1.31 $A=\{1,2\}$, $B=\{2\}$, $C=\{1,2,3\}$, $D=\{2,3\}$.

1.32 $A=\{1,2,3\}$, $B=\{1,2,4\}$, $C=\{1,3,4\}$, $D=\{2,3,4\}$.

1.33 $A=\{1\}$, $B=\{2\}$.

1.34 $A=\{1,2\}$, $B=\{2,3\}$.

1.35 Let $U=\{1,2,\ldots,8\}$, $A=\{1,2,3,5\}$, $B=\{1,2,4,6\}$ and $C=\{1,3,4,7\}$. See Figure 5.

Exercises for Section 1.4: Indexed Collections of Sets

1.36 $\bigcup_{\alpha\in A}S_\alpha=S_1\cup S_3\cup S_4=[0,3]\cup[2,5]\cup[3,6]=[0,6]$.
$\bigcap_{\alpha\in A}S_\alpha=S_1\cap S_3\cap S_4=\{3\}$.

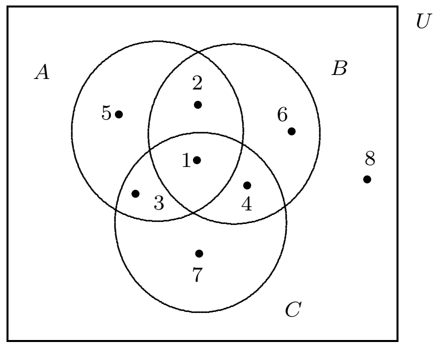

Figure 5: Answer for Exercise 1.35

1.37 $\bigcup_{X\in S} X = A \cup B \cup C = \{0,1,2,\ldots,5\}$ and $\bigcap_{X\in S} X = A \cap B \cap C = \{2\}$.

1.38 (a) $\bigcup_{\alpha\in S} A_\alpha = A_1 \cup A_2 \cup A_4 = \{1\} \cup \{4\} \cup \{16\} = \{1,4,16\}$.
$\bigcap_{\alpha\in S} A_\alpha = A_1 \cap A_2 \cap A_4 = \emptyset$.

(b) $\bigcup_{\alpha\in S} B_\alpha = B_1 \cup B_2 \cup B_4 = [0,2] \cup [1,3] \cup [3,5] = [0,5]$.
$\bigcap_{\alpha\in S} B_\alpha = B_1 \cap B_2 \cap B_4 = \emptyset$.

(c) $\bigcup_{\alpha\in S} C_\alpha = C_1 \cup C_2 \cup C_4 = (1,\infty) \cup (2,\infty) \cup (4,\infty) = (1,\infty)$.
$\bigcap_{\alpha\in S} C_\alpha = C_1 \cap C_2 \cap C_4 = (4,\infty)$.

1.39 Since $|A| = 26$ and $|A_\alpha| = 3$ for each $\alpha \in A$, we need to have at least nine sets of cardinality 3 for their union to be A; that is, in order for $\bigcup_{\alpha\in S} A_\alpha = A$, we must have $|S| \geq 9$. However, if we let $S = \{a,d,g,j,m,p,s,v,y\}$, then $\bigcup_{\alpha\in S} A_\alpha = A$. Hence the smallest cardinality of a set S with $\bigcup_{\alpha\in S} A_\alpha = A$ is 9.

1.40 (a) $\bigcup_{i=1}^{5} A_{2i} = A_2 \cup A_4 \cup A_6 \cup A_8 \cup A_{10} = \{1,3\} \cup \{3,5\} \cup \{5,7\} \cup \{7,9\} \cup \{9,11\} = \{1,3,5,\ldots,11\}$.

(b) $\bigcup_{i=1}^{5}(A_i \cap A_{i+1}) = \bigcup_{i=1}^{5}(\{i-1,i+1\} \cap \{i,i+2\}) = \bigcup_{i=1}^{5} \emptyset = \emptyset$.

(c) $\bigcup_{i=1}^{5}(A_{2i-1} \cap A_{2i+1}) = \bigcup_{i=1}^{5}(\{2i-2,2i\} \cap \{2i,2i+2\}) = \bigcup_{i=1}^{5}\{2i\} = \{2,4,6,8,10\}$.

1.41 (a) $\{A_n\}_{n\in\mathbf{N}}$, where $A_n = \{x \in \mathbf{R} : 0 \leq x \leq 1/n\} = [0,1/n]$.

(b) $\{A_n\}_{n\in\mathbf{N}}$, where $A_n = \{a \in \mathbf{Z} : |a| \leq n\} = \{-n,-(n-1),\ldots,(n-1),n\}$.

1.42 (a) $A_n = \left[1, 2+\frac{1}{n}\right)$, $\bigcup_{n\in\mathbf{N}} A_n = [1,3)$ and $\bigcap_{n\in\mathbf{N}} A_n = [1,2]$.

(b) $A_n = \left(-\frac{2n-1}{n}, 2n\right)$, $\bigcup_{n\in\mathbf{N}} A_n = (-2,\infty)$ and $\bigcap_{n\in\mathbf{N}} A_n = (-1,2)$.

1.43 $\bigcup_{r\in\mathbf{R}^+} A_r = \bigcup_{r\in\mathbf{R}^+}(-r,r) = \mathbf{R}$;
$\bigcap_{r\in\mathbf{R}^+} A_r = \bigcap_{r\in\mathbf{R}^+}(-r,r) = \{0\}$.

1.44 For $I = \{2,8\}$, $|\bigcup_{i\in I} A_i| = 8$. Observe that there is no set I such that $|\bigcup_{i\in I} A_i| = 10$, for in this case, we must have either two 5-element subsets of A or two 3-element subsets of A and a 4-element subset of A. In each case, not every two subsets are disjoint. Furthermore, there is no set I such that $|\bigcup_{i\in I} A_i| = 9$, for in this case, one must either have a 5-element subset of A and a 4-element subset of A (which are not disjoint) or three 3-element subsets of A. No 3-element subset of A contains 1 and only one such subset contains 2. Thus $4,5 \in I$ but there is no third element for I.

1.45 $\bigcup_{n\in\mathbf{N}} A_n = \bigcup_{n\in\mathbf{N}}(-\frac{1}{n}, 2-\frac{1}{n}) = (-1, 2)$;

$\bigcap_{n\in\mathbf{N}} A_n = \bigcap_{n\in\mathbf{N}}(-\frac{1}{n}, 2-\frac{1}{n}) = [0, 1]$.

Exercises for Section 1.5: Partitions of Sets

1.46 (a) S_1 is a partition of A.

(b) S_2 is not a partition of A because g belongs to no element of S_2.

(c) S_3 is a partition of A.

(d) S_4 is not a partition of A because $\emptyset \in S_4$.

(e) S_5 is not a partition of A because b belongs to two elements of S_5.

1.47 (a) S_1 is not a partition of A since 4 belongs to no element of S_1.

(b) S_2 is a partition of A.

(c) S_3 is not a partition of A because 2 belongs to two elements of S_3.

(d) S_4 is not a partition of A since S_4 is not a set of subsets of A.

1.48 $S = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$; $|S| = 3$.

1.49 $A = \{1, 2, 3, 4\}$. $S_1 = \{\{1\}, \{2\}, \{3, 4\}\}$ and $S_2 = \{\{1, 2\}, \{3\}, \{4\}\}$.

1.50 Let $S = \{A_1, A_2, A_3\}$, where $A_1 = \{x \in \mathbf{N} : x > 5\}$, $A_2 = \{x \in \mathbf{N} : x < 5\}$ and $A_3 = \{5\}$.

1.51 Let $S = \{A_1, A_2, A_3\}$, where $A_1 = \{x \in \mathbf{Q} : x > 1\}$, $A_2 = \{x \in \mathbf{Q} : x < 1\}$ and $A_3 = \{1\}$.

1.52 $A = \{1, 2, 3, 4\}$, $S_1 = \{\{1\}, \{2\}, \{3, 4\}\}$ and $S_2 = \{\{\{1\}, \{2\}\}, \{\{3, 4\}\}\}$.

1.53 Let $S = \{A_1, A_2, A_3, A_4\}$, where

$A_1 = \{x \in \mathbf{Z} : x \text{ is odd and } x \text{ is positive}\}$,

$A_2 = \{x \in \mathbf{Z} : x \text{ is odd and } x \text{ is negative}\}$,

$A_3 = \{x \in \mathbf{Z} : x \text{ is even and } x \text{ is nonnegative}\}$,

$A_4 = \{x \in \mathbf{Z} : x \text{ is even and } x \text{ is negative}\}$.

1.54 Let $S = \{\{1\}, \{2\}, \{3, 4, 5, 6\}, \{7, 8, 9, 10\}, \{11, 12\}\}$ and $T = \{\{1\}, \{2\}, \{3, 4, 5, 6\}, \{7, 8, 9, 10\}\}$.

1.55 $|\mathcal{P}_1| = 2$, $|\mathcal{P}_2| = 3$, $|\mathcal{P}_3| = 5$, $|\mathcal{P}_4| = 8$, $|\mathcal{P}_5| = 13$, $|\mathcal{P}_6| = 21$.

1.56 (a) Suppose that a collection S of subsets of A satisfies Definition 1. Then every subset is nonempty. Every element of A belongs to a subset in S. If some element $a \in A$ belonged to more than one subset, then the subsets in S would not be pairwise disjoint. So the collection satisfies Definition 2.

(b) Suppose that a collection S of subsets of A satisfies Definition 2. Then every subset is nonempty and (1) in Definition 3 is satisfied. If two subsets A_1 and A_2 in S were neither equal nor disjoint, then $A_1 \neq A_2$ and there is an element $a \in A$ such that $a \in A_1 \cap A_2$, which would not satisfy Definition 2. So condition (2) in Definition 3 is satisfied. Since every element of A belongs to a (unique) subset in S, condition (3) in Definition 3 is satisfied. Thus Definition 3 itself is satisfied.

(c) Suppose that a collection S of subsets of A satisfies Definition 3. By condition (1) in Definition 3, every subset is nonempty. By condition (2), the subsets are pairwise disjoint. By condition (3), every element of A belongs to a subset in S. So Definition 1 is satisfied.

Exercises for Section 1.6: Cartesian Products of Sets

1.57 $A \times B = \{(x,x),(x,y),(y,x),(y,y),(z,x),(z,y)\}$.

1.58 $A \times A = \{(1, 1), (1, \{1\}), (1, \{\{1\}\}), (\{1\}, 1), (\{1\}, \{1\}), (\{1\}, \{\{1\}\}), (\{\{1\}\}, 1), (\{\{1\}\}, \{1\}), (\{\{1\}\}, \{\{1\}\})\}$.

1.59 $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, A\}$,

$A \times \mathcal{P}(A) = \{(a,\emptyset),(a,\{a\}),(a,\{b\}),(a,A),(b,\emptyset),(b,\{a\}),(b,\{b\}),(b,A)\}$.

1.60 $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, A\}$,

$A \times \mathcal{P}(A) = \{(\emptyset,\emptyset),(\emptyset,\{\emptyset\}),(\emptyset,\{\{\emptyset\}\}),(\emptyset,A),(\{\emptyset\},\emptyset),(\{\emptyset\},\{\emptyset\}),(\{\emptyset\},\{\{\emptyset\}\}),(\{\emptyset\},A)\}$.

1.61 $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, A\}$, $\mathcal{P}(B) = \{\emptyset, B\}$, $A \times B = \{(1,\emptyset),(2,\emptyset)\}$,

$\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset,\emptyset),(\emptyset,B),(\{1\},\emptyset),(\{1\},B),(\{2\},\emptyset),(\{2\},B),(A,\emptyset),(A,B)\}$.

1.62 $\{(x,y): \ x^2+y^2=4\}$, which is a circle centered at (0, 0) with radius 2.

1.63 $S = \{(3,0),(2,1),(2,-1),(1,2),(1,-2),(0,3),(0,-3),(-3,0),(-2,1),(-2,-1),(-1,2),(-1,-2)\}$. See Figure 6.

1.64 $A \times B = \{(1,1),(2,1)\}$,

$\mathcal{P}(A \times B) = \{\emptyset, \{(1,1)\}, \{(2,1)\}, A \times B\}$

1.65 $A = \{x \in \mathbf{R} : |x-1| \leq 2\} = \{x \in \mathbf{R} : -1 \leq x \leq 3\} = [-1,3]$

$B = \{y \in \mathbf{R} : |y-4| \leq 2\} = \{y \in \mathbf{R} : 2 \leq y \leq 6\} = [2,6]$,

$A \times B = [-1,3] \times [2,6]$, which is the set of all points on and within the square bounded by $x=-1$, $x=3$, $y=2$ and $y=6$.

1.66 $A = \{a \in \mathbf{R} : |a| \leq 1\} = \{a \in \mathbf{R} : -1 \leq a \leq 1\} = [-1,1]$

$B = \{b \in \mathbf{R} : |b| = 1\} = \{-1,1\}$,

$A \times B$ is the set of all points (x,y) on the lines $y=1$ or $y=-1$ with $x \in [-1,1]$, while $B \times A$ is the set of all points (x,y) on the lines $x=1$ or $x=-1$ with $y \in [-1,1]$. Therefore, $(A \times B) \cup (B \times A)$

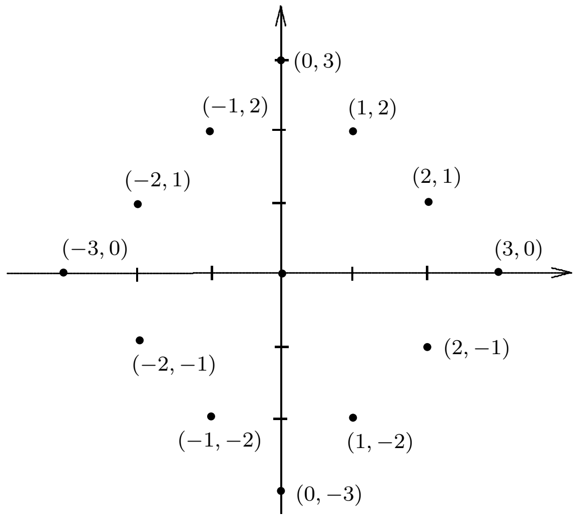

Figure 6: Answer for Exercise 1.63

is the set of all points lying on (but not within) the square bounded by $x = 1$, $x = -1$, $y = 1$ and $y = -1$.

Additional Exercises for Chapter 1

1.67 (a) $A = \{4k + 3 : k \in \mathbf{Z}\} = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

(b) $B = \{5k - 1 : k \in \mathbf{Z}\} = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$.

1.68 (a) $A = \{x \in S : \ |x| \geq 1\} = \{x \in S : \ x \neq 0\}$.

(b) $B = \{x \in S : \ x \leq 0\}$.

(c) $C = \{x \in S : \ -5 \leq x \leq 7\} = \{x \in S : \ |x - 1| \leq 6\}$.

(d) $D = \{x \in S : \ x \neq 5\}$.

1.69 (a) $\{0, 2, -2\}$ (b) $\{ \}$ (c) $\{3, 4, 5\}$ (d) $\{1, 2, 3\}$

(e) $\{-2, 2\}$ (f) $\{ \}$ (g) $\{-3, -2, -1, 1, 2, 3\}$.

1.70 (a) $|A| = 6$ (b) $|B| = 0$ (c) $|C| = 3$

(d) $|D| = 0$ (e) $|E| = 10$ (f) $|F| = 20$.

1.71 $A \times B = \{(-1, x), (-1, y), (0, x), (0, y), (1, x), (1, y)\}$.

1.72 (a) $(A \cup B) - (B \cap C) = \{1, 2, 3\} - \{3\} = \{1, 2\}$.

(b) $\overline{A} = \{3\}$.

(c) $\overline{B \cup C} = \overline{\{1, 2, 3\}} = \emptyset$.

(d) $A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$.

1.73 Let $S = \{\{1\}, \{2\}, \{3,4\}, A\}$ and let $B = \{3,4\}$.

1.74 $\mathcal{P}(A) = \{\emptyset, \{1\}\}$, $\mathcal{P}(C) = \{\emptyset, \{1\}, \{2\}, C\}$. Let $B = \{\emptyset, \{1\}, \{2\}\}$.

1.75 Let $A = \{\emptyset\}$ and $B = \mathcal{P}(A) = \{\emptyset, \{\emptyset\}\}$.

1.76 Only $B = C = \emptyset$ and $D = E$.

1.77 $U = \{1,2,3,5,7,8,9\}$, $A = \{1,2,5,7\}$ and $B = \{5,7,8\}$.

1.78 (a) A_r is the set of all points in the plane lying on the circle $x^2 + y^2 = r^2$.
$\bigcup_{r \in I} A_r = \mathbf{R} \times \mathbf{R}$ (the plane) and $\bigcap_{r \in I} A_r = \emptyset$.

(b) B_r is the set of all points lying on and inside the circle $x^2 + y^2 = r^2$.
$\bigcup_{r \in I} B_r = \mathbf{R} \times \mathbf{R}$ and $\bigcap_{r \in I} B_r = \{(0,0)\}$.

(c) C_r is the set of all points lying outside the circle $x^2 + y^2 = r^2$.
$\bigcup_{r \in I} C_r = \mathbf{R} \times \mathbf{R} - \{(0,0)\}$ and $\bigcap_{r \in I} C_r = \emptyset$.

1.79 Let $A_1 = \{1,2,3,4\}$, $A_2 = \{3,5,6\}$, $A_3 = \{1,3\}$, $A_4 = \{1,2,4,5,6\}$. Then $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_3 \cap A_4| = 1$, $|A_1 \cap A_3| = |A_2 \cap A_4| = 2$ and $|A_1 \cap A_4| = 3$.

1.80 (a) (i) Give an example of five sets A_i $(1 \le i \le 5)$ such that $|A_i \cap A_j| = |i - j|$ for every two integers i and j with $1 \le i < j \le 5$.

(ii) Determine the minimum positive integer k such that there exist four sets A_i $(1 \le i \le 4)$ satisfying the conditions of Exercise 1.79 and $|A_1 \cup A_2 \cup A_3 \cup A_4| = k$.

(b) (i) $A_1 = \{1,2,3,4,7,8,9,10\}$
$A_2 = \{3,5,6,11,12,13\}$
$A_3 = \{1,3,14,15\}$
$A_4 = \{1,2,4,5,6,16\}$
$A_5 = \{7,8,9,10,11,12,13,14,15,16\}$.

(ii) The minimum positive integer k is 5. The example below shows that $k \le 5$.

Let $A_1 = \{1,2,3,4\}$, $A_2 = \{1,5\}$, $A_3 = \{1,4\}$, $A_4 = \{1,2,3,5\}$.

If $k = 4$, then since $|A_1 \cap A_4| = 3$, A_1 and A_4 have exactly three elements in common, say 1, 2, 3. So each of A_1 and A_4 is either $\{1,2,3\}$ or $\{1,2,3,4\}$. They cannot both be $\{1,2,3,4\}$. Also, they cannot both be $\{1,2,3\}$ because A_3 would have to contain two of 1, 2, 3 and so $|A_3 \cap A_4| \ge 2$, which is not true. So we can assume that $A_1 = \{1,2,3,4\}$ and $A_4 = \{1,2,3\}$. However, A_2 must contain two of 1, 2, 3 and so $|A_1 \cap A_2| \ge 2$, which is impossible.

1.81 (a) $|S| = |T| = 10$.

(b) $|S| = |T| = 5$.

(c) $|S| = |T| = 6$.

1.82 Let $A = \{1,2,3,4\}$, $A_1 = \{1,2\}$, $A_2 = \{1,3\}$, $A_3 = \{3,4\}$. These examples show that $k \le 4$. Since $|A_1 - A_3| = |A_3 - A_1| = 2$, it follows that A_1 contains two elements not in A_3, while A_3 contains two elements not in A_1. Thus $|A| \ge 4$ and so $k = 4$ is the smallest positive integer with this property.

1.83 (a) $S = \{(-3, 4), (0, 5), (3, 4), (4, 3)\}$.

(b) $C = \{a \in B : (a, b) \in S\} = \{3, 4\}$

$D = \{b \in A : (a, b) \in S\} = \{3, 4\}$

$C \times D = \{(3, 3), (3, 4), (4, 3), (4, 3)\}$.

1.84 $A = \{1, 2, 3\}$, $B = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, $C = \{\{1\}, \{2\}, \{3\}\}$.

$D = \mathcal{P}(C) = \{\emptyset, \{\{1\}\}, \{\{2\}\}, \{\{3\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{3\}\}, \{\{2\}, \{3\}\}, C\}$.

1.85 $S = \{x \in \mathbf{R} : x^2 + 2x - 1 = 0\} = \{-1 + \sqrt{2}, -1 - \sqrt{2}\}$.

$A_{-1+\sqrt{2}} = \{-1 + \sqrt{2}, \sqrt{2}\}$, $A_{-1-\sqrt{2}} = \{-1 - \sqrt{2} - \sqrt{2}\}$.

(a) $A_s = A_{-1-\sqrt{2}}$ and $A_t = A_{-1+\sqrt{2}}$.

$A_s \times A_t = \{(-1 - \sqrt{2}, -1 + \sqrt{2}), (-1 - \sqrt{2}, \sqrt{2}), (-\sqrt{2}, 1 + \sqrt{2}), (-\sqrt{2}, \sqrt{2})\}$.

(b) $C = \{ab : (a, b) \in B\} = \{-1, -\sqrt{2} - 2, \sqrt{2} - 2, -2\}$. The sum of the elements in C is -7.