Chapter 1

First-Order Differential Equations

1.1 Terminology and Separable Equations

1. The differential equation is separable because it can be written

$$3y^2\frac{dy}{dx} = 4x,$$

or, in differential form,

$$3y^2\,dy = 4x\,dx.$$

Integrate to obtain

$$y^3 = 2x^2 + k.$$

This implicitly defines a general solution, which can be written explicitly as

$$y = (2x^2 + k)^{1/3},$$

with k an arbitrary constant.

2. Write the differential equation as

$$x\frac{dy}{dx} = -y,$$

which separates as

$$\frac{1}{y}\,dy = -\frac{1}{x}\,dx$$

if $x \neq 0$ and $y \neq 0$. Integrate to get

$$\ln|y| = -\ln|x| + k.$$

Then $\ln|xy| = k$, so

$$xy = c$$

with c constant ($c = e^k$). $y = 0$ is a singular solution, satisfying the original differential equation.

3. If $\cos(y) \neq 0$, the differential equation is

$$\begin{aligned}\frac{y}{dx} &= \frac{\sin(x+y)}{\cos(y)} \\ &= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(y)} \\ &= \sin(x) + \cos(x)\tan(y).\end{aligned}$$

   There is no way to separate the variables in this equation, so the differential equation is not separable.

4. Write the differential equation as

$$e^x e^y \frac{dy}{dx} = 3x,$$

   which separates in differential form as

$$e^y\,dy = 3xe^{-x}\,dx.$$

   Integrate to get

$$e^y = -3e^{-x}(x+1) + c,$$

   with c constant. This implicitly defines a general solution.

5. The differential equation can be written

$$x\frac{dy}{dx} = y^2 - y,$$

   or

$$\frac{1}{y(y-1)}\,dy = \frac{1}{x}\,dx,$$

   and is therefore separable. Separating the variables assumes that $y \neq 0$ and $y \neq 1$. We can further write

$$\left(\frac{1}{y-1} - \frac{1}{y}\right)dy = \frac{1}{x}\,dx.$$

   Integrate to obtain

$$\ln|y-1| - \ln|y| = \ln|x| + k.$$

   Using properties of the logarithm, this is

$$\ln\left|\frac{y-1}{xy}\right| = k.$$

Then

$$\frac{y-1}{xy} = c,$$

with $c = e^k$ constant. Solve this for y to obtain the general solution

$$y = \frac{1}{1-cx}.$$

$y = 0$ and $y = 1$ are singular solutions because these satisfy the differential equation, but were excluded in the algebra of separating the variables.

6. The differential equation is not separable.

7. The equation is separable because it can be written in differential form as

$$\frac{\sin(y)}{\cos(y)}\,dy = \frac{1}{x}\,dx.$$

This assumes that $x \neq 0$ and $\cos(y) \neq 0$. Integrate this equation to obtain

$$-\ln|\cos(y)| = \ln|x| + k.$$

This implicitly defines a general solution. From this we can also write

$$\sec(y) = cx$$

with c constant.

The algebra of separating the variables required that $\cos(y) \neq 0$. Now $\cos(y) = 0$ if $y = (2n+1)\pi/2$, with n any integer. Now $y = (2n+1)\pi/2$ also satisfies the original differential equation, so these are singular solutions.

8. The differential equation itself requires that $y \neq 0$ and $x \neq -1$. Write the equation as

$$\frac{x}{y}\frac{dy}{dx} = \frac{2y^2+1}{x}$$

and separate the variables to get

$$\frac{1}{y(2y^2+1)}\,dy = \frac{1}{x(x+1)}\,dx.$$

Use a partial fractions decomposition to write this as

$$\left(\frac{1}{y} - \frac{2y}{2y^2+1}\right)dy = \left(\frac{1}{x} - \frac{1}{x+1}\right)dx.$$

Integrate to obtain

$$\ln|y| - \frac{1}{2}\ln(1+2y^2) = \ln|x| - \ln|x+1| + c$$

with c constant. This implicitly defines a general solution. We can go a step further by writing this equation as

$$\ln\left(\frac{y}{\sqrt{1+2y^2}}\right) = \ln\left(\frac{x}{x+1}\right) + c$$

and take the exponential of both sides to get

$$\frac{y}{\sqrt{1+2y^2}} = k\left(\frac{x}{x+1}\right),$$

which also defines a general solution.

9. The differential equation is

$$\frac{dy}{dx} = e^x - y + \sin(y),$$

and this is not separable. It is not possible to separate all terms involving x on one side of the equation and all terms involving y on the other.

10. Substitute

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y),$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

and

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

into the differential equation to get the separated differential form

$$(\cos(y) - \sin(y))\,dy = (\cos(x) - \sin(x))\,dx.$$

Integrate to obtain the implicitly defined general solution

$$\cos(y) + \sin(y) = \cos(x) + \sin(x) + c.$$

11. If $y \neq -1$ and $x \neq 0$, we obtain the separated equation

$$\frac{y^2}{y+1}\,dy = \frac{1}{x}\,dx.$$

To make the integration easier, write this as

$$\left(y - 1 + \frac{1}{1+y}\right)dy = \frac{1}{x}\,dx.$$

Integrate to obtain

$$\frac{1}{2}y^2 - y + \ln|1+y| = \ln|x| + c.$$

This implicitly defines a general solution. The initial condition is $y(3e^2) = 2$, so put $y = 2$ and $x = 3e^2$ to obtain

$$2 - 2 + \ln(3) = \ln(3e^2) + c.$$

Now

$$\ln(3e^2) = \ln(3) + \ln(e^2) = \ln(3) + 2,$$

so

$$\ln(3) = \ln(3) + 2 + c.$$

Then $c = -2$ and the solution of the initial value problem is implicitly defined by

$$\frac{1}{2}y^2 - y + \ln|1 + y| = \ln|x| - 2.$$

12. Integrate

$$\frac{1}{y+2}\,dy = 3x^2\,dx,$$

assuming that $y \neq -2$, to obtain

$$\ln|2 + y| = x^3 + c.$$

This implicitly defines a general solution. To have $y(2) = 8$, let $x = 2$ and $y = 8$ to obtain

$$\ln(10) = 8 + c.$$

The solution of the initial value problem is implicitly defined by

$$\ln|2 + y| = x^3 + \ln(10) - 8.$$

We can take this a step further and write

$$\ln\left(\frac{2+y}{10}\right) = x^3 - 8.$$

By taking the exponential of both sides of this equation we obtain the explicit solution

$$y = 10e^{x^3-8} - 2.$$

13. With $\ln(y^x) = x\ln(y)$, we obtain the separated equation

$$\frac{\ln(y)}{y}\,dy = 3x\,dx.$$

Integrate to obtain

$$(\ln(y))^2 = 3x^2 + c.$$

For $y(2) = e^3$, we need

$$(\ln(e^3))^2 = 3(4) + c,$$

or $9 = 12 + c$. Then $c = -3$ and the solution of the initial value problem is defined by

$$(\ln(y))^2 = 3x^2 - 3.$$

Solve this to obtain the explicit solution

$$y = e^{\sqrt{3(x^2-1)}}$$

if $|x| > 1$.

14. Because $e^{x-y^2} = e^x e^{-y^2}$, the variables can be separated to obtain

$$2ye^{y^2}\,dy = e^x\,dx.$$

Integrate to get

$$e^{y^2} = e^x + c.$$

To satisfy $y(4) = -2$ we need

$$e^4 = e^4 + c$$

so $c = 0$ and the solution of the initial value problem is implicitly defined by

$$e^{y^2} = e^x,$$

which reduces to the simpler equation

$$x = y^2.$$

Because $y(4) = -2$, the explicit solution is $y = -\sqrt{x}$ for $x > 0$.

15. Separate the variables to obtain

$$y\cos(3y)\,dy = 2x\,dx.$$

Integrate to get

$$\frac{1}{3}y\sin(3y) + \frac{1}{9}\cos(3y) = x^2 + c,$$

which implicitly defines a general solution. For $y(2/3) = \pi/3$, we need

$$\frac{1}{3}\frac{\pi}{3}\sin(\pi) + \frac{1}{9}\cos(\pi) = \frac{4}{9} + c.$$

This reduces to

$$-\frac{1}{9} = \frac{4}{9} + c,$$

so $c = -5/9$ and the solution of the initial value problem is implicitly defined by

$$\frac{1}{3}y\sin(3y) + \frac{1}{9}\cos(3y) = x^2 - \frac{5}{9},$$

or

$$3y\sin(3y) + \cos(3y) = 9x^2 - 1.$$