Chapter 1

Stochastic Calculus for Finance I: The Binomial Asset Pricing Model

1.1 The Binomial No-Arbitrage Pricing Model

1.1.

Proof. If we get the up sate, then $X_1 = X_1(H) = \Delta_0 uS_0 + (1+r)(X_0 - \Delta_0 S_0)$; if we get the down state, then $X_1 = X_1(T) = \Delta_0 dS_0 + (1+r)(X_0 - \Delta_0 S_0)$. If X_1 has a positive probability of being strictly positive, then we must either have $X_1(H) > 0$ or $X_1(T) > 0$.

(i) If $X_1(H) > 0$, then $\Delta_0 uS_0 + (1+r)(X_0 - \Delta_0 S_0) > 0$. Plug in $X_0 = 0$, we get $u\Delta_0 > (1+r)\Delta_0$. By condition $d < 1+r < u$, we conclude $\Delta_0 > 0$. In this case, $X_1(T) = \Delta_0 dS_0 + (1+r)(X_0 - \Delta_0 S_0) = \Delta_0 S_0[d-(1+r)] < 0$.

(ii) If $X_1(T) > 0$, then we can similarly deduce $\Delta_0 < 0$ and hence $X_1(H) < 0$.

So we cannot have X_1 strictly positive with positive probability unless X_1 is strictly negative with positive probability as well, regardless the choice of the number Δ_0.

Remark: Here the condition $X_0 = 0$ is not essential, as far as a property definition of arbitrage for arbitrary X_0 can be given. Indeed, for the one-period binomial model, we can define arbitrage as a trading strategy such that $P(X_1 \geq X_0(1+r)) = 1$ and $P(X_1 > X_0(1+r)) > 0$. First, this is a generalization of the case $X_0 = 0$; second, it is "proper" because it is comparing the result of an arbitrary investment involving money and stock markets with that of a safe investment involving only money market. This can also be seen by regarding X_0 as borrowed from money market account. Then at time 1, we have to pay back $X_0(1+r)$ to the money market account. In summary, arbitrage is a trading strategy that beats "safe" investment.

Accordingly, we revise the proof of Exercise 1.1. as follows. If X_1 has a positive probability of being strictly larger than $X_0(1+r)$, the either $X_1(H) > X_0(1+r)$ or $X_1(T) > X_0(1+r)$. The first case yields $\Delta_0 S_0(u-1-r) > 0$, i.e. $\Delta_0 > 0$. So $X_1(T) = (1+r)X_0 + \Delta_0 S_0(d-1-r) < (1+r)X_0$. The second case can be similarly analyzed. Hence we cannot have X_1 strictly greater than $X_0(1+r)$ with positive probability unless X_1 is strictly smaller than $X_0(1+r)$ with positive probability as well.

Finally, we comment that the above formulation of arbitrage is equivalent to the one in the textbook. For details, see Shreve [7], Exercise 5.7. □

1.2.

Proof. $X_1(u) = \Delta_0 \times 8 + \Gamma_0 \times 3 - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0) = 3\Delta_0 + 1.5\Gamma_0$, and $X_1(d) = \Delta_0 \times 2 - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0) = -3\Delta_0 - 1.5\Gamma_0$. That is, $X_1(u) = -X_1(d)$. So if there is a positive probability that X_1 is positive, then there is a positive probability that X_1 is negative.

Remark: Note the above relation $X_1(u) = -X_1(d)$ is not a coincidence. In general, let V_1 denote the payoff of the derivative security at time 1. Suppose $\bar{X}_0$ and $\bar{\Delta}_0$ are chosen in such a way that V_1 can be